CHARACTERIZATIONS OF ARITHMETICAL PROGRESSION SERIES WITH SOME COUNTEREXAMPLES ON INTERPOLATION

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Abstract. Characterizations of the functions

$$f(z) = \frac{z}{(1-z)^2} \text{ and } f(z) = \frac{1+z}{(1-z)^2}$$

are given. We also give counterexamples to show that some generalized problems on interpolation do not hold.

1. Introduction. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a power series with positive coefficients and positive radius of convergence. Associate $a_{-1} = 0$, $r_n = a_{n-1}/a_n$ for $n = 0, 1, 2, \ldots$. It is known [3] that if

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = R,$$

then the radius of convergence of

$$\sum_{n=1}^{\infty} a_n z^n$$

is R and also [2] if

$$\liminf_{n\to\infty} \frac{\log(\frac{a_n}{a_{n+1}})}{\log n} = L > 0,$$

then

$$\sum_{n=0}^{\infty} a_n z^n$$

is an entire function of order $\leq 1/L$. In addition, if

$$\lim_{n\to\infty} \frac{\log(\frac{a_n}{a_{n+1}})}{\log n} = L > 0,$$

then

$$\sum_{n=0}^{\infty} a_n z^n$$

is of order $1/L$.

In his paper [1], Abi-Khuzam used the term "normalize" as follows. If $g(z) = f(cz)$ with c a positive constant, then $g(z)$ will have positive coefficients $\{b_n\}$ and if $s_n = b_{n-1}/b_n$, then $g(s_n) = f(r_n)$, $n = 0, 1, 2, \ldots$. Thus, one can normalize the function $f(z)$ to make a_1 equal to a given number without changing the sequence $\{f(r_n)\}$ and such a normalization was incorporated in [1] where the following theorems were proven.

Theorem 1 [1]. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a power series such that

(i) $a_n > 0$ for $n = 0, 1, 2, \ldots,$
(ii) $0 < r_n < R < \infty$, where R is the radius of convergence, and
(iii) there exists a positive real number α such that $a_1 = \alpha + 1$ and

$$f(r_n) = \left(\frac{n+\alpha}{\alpha}\right)^{\alpha+1} \text{ for } n = 0, 1, 2, \ldots,$$

then

$$f(z) = (1-z)^{-\alpha-1} \text{ for all } |z| < 1.$$

Theorem 2 [1]. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an entire function such that

(i) $a_n > 0$ for $n = 0, 1, 2, \ldots$, and
(ii) $a_1 = 1$ and $f(r_n) = e^n$ for $n = 0, 1, 2, \ldots$,

then

$$f(z) = e^z \text{ for all } z.$$

In concluding an interesting paper, Abi-Khuzam [1] raised the following questions.

Problem 1. If

$$f(r) = \sum_{n=0}^{\infty} a_n r^n, \quad g(r) = \sum_{n=0}^{\infty} b_n r^n,$$

$r_n = a_{n-1}/a_n$, $s_n = b_{n-1}/b_n$, and $f(r_n) = g(s_n)$ for all $n \geq 0$, does it follow that modulo normalization $f = g$?

Problem 2. Since the hypothesis of Theorem 2 holds for functions with positive and negative coefficients e.g., $f(z) = e^{-z}$, one may ask whether it holds true under the assumption $a_n \neq 0$, instead of $a_n > 0$.

In this paper we will find characterizations of a type similar to Theorems 1 and 2 and also solve Problems 1 and 2.

1. Characterization of $\mathbf{f(z) = z/(1-z)^2}$. Consider the power series

$$f(x) = x + 2x^2 + 3x^3 + \cdots = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad -1 < x < 1.$$

Here $r_n = n/n+1$ for $n = 0, 1, 2, \ldots$. Since $0 \leq r_n < 1$, $f(r_n) = n(n+1)$, $n = 0, 1, 2, \ldots$. This suggests the following theorem.

Theorem 3. If

$$f(z) = \sum_{n=1}^{\infty} b_n z^n$$

is a power series such that

(i) $b_n > 0$ for $n = 1, 2, \ldots$.
(ii) $0 \leq s_n < R < \infty$, $n = 0, 1, 2, \ldots$, where R is the radius of convergence and $s_n = b_n/b_{n+1}$, and
(iii) $b_1 = 1$, $b_2 = 2$, and $f(s_n) = n(n+1)$,

then

$$f(z) = \frac{z}{(1-z)^2} \text{ for all } |z| < 1.$$

Proof. It is clear that

$$f(z) = \frac{z}{(1-z)^2}$$

satisfies (i)–(iii). We now prove that

$$\frac{z}{(1-z)^2}$$

is the only such function. This is clearly true if $z = 0$. Suppose now that $z \neq 0$. Let

$$P(z) = \frac{f(z)}{z}.$$

Then

$$P(z) = \sum_{n=0}^{\infty} b_{n+1} z^n$$

with the same sequence $\{s_n\}$ associated. Using $b_1 = 1$ and $b_2 = 2$ we can write

$$P(z) = 1 + 2z + \sum_{n=2}^{\infty} b_{n+1} z^n.$$

Letting $a_n = b_{n+1}$, $n = 0, 1, 2, \dots$ we have

$$P(z) = 1 + 2z + \sum_{n=2}^{\infty} a_n z^n$$

with the sequence

$$\{r_n\} = \left\{ \frac{a_{n-1}}{a_n} \right\}$$

associated. However, $P(r_n) = P(s_n)$ for $n = 0, 1, 2, \dots$. Then

$$P(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } a_n > 0 \text{ for } n = 0, 1, 2, \dots,$$

$$0 < r_n = \frac{a_{n-1}}{a_n} = \frac{b_n}{b_{n+1}} = s_n < \infty, \quad a_1 = 2, \text{ and}$$

$$P(r_n) = P(s_n) = \frac{f(s_n)}{s_n}.$$

Next, we show by induction that $s_n = n/n+1$. From the representation of $P(z)$ above, $s_1 = 1/2$. Suppose then that $s_n = n/n+1$. We shall show that $s_{n+1} = (n+1)/(n+2)$. Now $s_1 s_2 \dots s_n = 1/b_{n+1}$. Thus,

$$b_{n+2} = \frac{1}{s_1 s_2 \dots s_n s_{n+1}} = \frac{1}{\frac{1}{2}\frac{2}{3} \cdots \frac{n}{n+1} s_{n+1}}.$$

Therefore,

$$b_{n+2} = \frac{n+1}{s_{n+1}} \text{ or } s_{n+1} = \frac{n+1}{b_{n+2}} = \frac{b_{n+1}}{b_{n+2}}$$

which implies that $b_{n+1} = n+1$ and consequently $s_{n+1} = (n+1)/(n+2)$.

As a result we get

$$P(r_n) = \frac{n(n+1)}{\frac{n}{n+1}} = (n+1)^2$$

for $n = 0, 1, 2, \ldots$. Thus, by Theorem 1 (with $\alpha = 1$) it follows that

$$P(z) = \frac{1}{(1-z)^2.}$$

Consequently,

$$f(z) = \frac{z}{(1-z)^2}$$

and the proof of Theorem 3 is complete.

3. Characterization of $\mathbf{f(z) = (1+z)/(1-z)^2}$. To find such a characterization we will use the following.

Lemma. The function

$$f(x) = \left(\frac{x}{x-1}\right)^{2x-1}$$

is a decreasing function on $[2, \infty)$.

Proof. $\log f(x) = (2x-1)[\log x - \log(x-1)]$ implies

$$\frac{f'(x)}{f(x)} = 2\log\frac{x}{x-1} - \frac{2x-1}{x(x-1)}$$

and it suffices to show that

$$\log\frac{x}{x-1} \leq \frac{2x-1}{2x(x-1)}.$$

But this follows easily using the derivative of

$$\log \frac{x}{x-1} - \frac{2x-1}{2x(x-1)}$$

and the proof of the lemma is complete.

Next consider the power series

$$f(x) = 1 + 3x + 5x^2 + \cdots = \sum_{n=0}^{\infty} (2n+1)x^n = \frac{1+x}{(1-x)^2}, \quad -1 < x < 1.$$

Here $r_0 = 0$ and $r_n = (2n-1)/(2n+1)$ for $n = 1, 2, \ldots$. Since $0 \leq r_n < 1$ for $n = 0, 1, 2, \ldots$, $f(r_0) = 1$, and $f(r_n) = n(2n+1)$, $n = 1, 2, \ldots$. This suggests the following theorem.

Theorem 4. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a power series such that

(i) $a_n > 0$ for $n = 0, 1, 2, \ldots$
(ii) $0 < r_n < R < \infty$, $n = 0, 1, 2, \ldots$, where R is the radius of convergence and $r_n = a_{n-1}/a_n$, and
(iii) $a_0 = 1$, $a_1 = 3$, and $f(r_n) = n(2n+1)$, $n = 1, 2, \ldots$

then

$$f(z) = \frac{1+z}{(1-z)^2} \text{ for all } |z| < 1.$$

Proof. Clearly

$$\frac{1+z}{(1-z)^2}$$

satisfies (i)–(iii). We now prove that

$$\frac{1+z}{(1-z)^2}$$

is the only such function. Using $a_0 = 1$ and $a_1 = 3$ we can write

$$f(z) = 1 + 3z + \sum_{n=2}^{\infty} a_n z^n$$

with the sequence

$$\{r_n\} = \left\{ \frac{a_{n-1}}{a_n} \right\}$$

associated. Now $f(r_1) = 3$. Since $r_1 = 1/3$, we have $3 = f(1/3)$. The positivity of $\{a_n\}$ implies f is increasing on $[0, R)$. By (iii), $\{f(r_n)\}$ is increasing. So $\{r_n\}$ is increasing. Thus, $r_1 \leq r_n < R$ or

$$\frac{1}{3} \leq r_n < R \text{ for } n = 1, 2, \dots .$$

Consequently,

$$\lim_{n\to\infty} r_n = R.$$

Let

$$\beta = \inf_{n\geq 1} \frac{2n+1}{2n-1} r_n \text{ and } \delta = \sup_{n\geq 1} \frac{2n+1}{2n-1} r_n.$$

Clearly

$$\frac{1}{3} \leq \beta \leq R \leq \delta \leq 3R.$$

Moreover,

$$\frac{2n+1}{\delta} r_n \leq 2n-1 \leq \frac{2n+1}{\beta} r_n, \text{ for } n = 1, 2, \dots .$$

In particular ($n = 1$) we see that $\beta \leq 1$ and $\delta \geq 1$. Let r be a number such that $0 < r < \beta$. We shall show by induction that

$$f^{(n)}(r) \leq \frac{(2n+1+\frac{\beta}{r}) \cdots (5+\frac{\beta}{r})(3+\frac{\beta}{r})}{2^n(\beta - r)^n} f(r), \text{ for } n = 1, 2, \dots .$$

First

$$f^{'}(r) = \sum_{n=1}^{\infty} na_n r^{n-1} = \frac{1}{2}\sum_{n=1}^{\infty} 2na_n r^{n-1} = \frac{1}{2}\sum_{n=1}^{\infty}(2n-1)a_n r^{n-1} + \frac{1}{2}\sum_{n=1}^{\infty} a_n r^{n-1}$$

$$\leq \frac{1}{2\beta}\sum_{n=1}^{\infty}(2n+1)a_{n-1}r^{n-1} + \frac{1}{2}\frac{f(r)}{r} = \frac{1}{2\beta}\sum_{n=0}^{\infty}(2n+3)a_n r^n + \frac{1}{2}\frac{f(r)}{r}$$

$$= \frac{1}{\beta}\sum_{n=0}^{\infty} na_n r^n + (\frac{3}{2\beta} + \frac{1}{2r})f(r) = \frac{r}{\beta}f^{'}(r) + (\frac{3}{2\beta} + \frac{1}{2r})f(r).$$

Thus,

$$f^{'}(r) \leq \frac{1}{2}\frac{3+\frac{\beta}{r}}{\beta - r}f(r). \tag{1}$$

Assume the induction hypothesis that

$$f^{(k-1)}(r) = \sum_{n=k-1}^{\infty} n(n-1)\cdots(n-k+2)a_n r^{n-k+1}$$

$$\leq \frac{(2k-1+\frac{\beta}{r})\cdots(5+\frac{\beta}{r})(3+\frac{\beta}{r})}{2^{k-1}(\beta-r)^{k-1}}f(r)$$

for some integer $k \geq 2$. Then

$$\begin{aligned}
f^{(k)}(r) &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n r^{n-k} \\
&= \sum_{n=1}^{\infty} (n+k-1)(n+k-2)\cdots n a_{n+k-1} r^{n-1} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} (2n+2k-2)(n+k-2)\cdots n a_{n+k-1} r^{n-1} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} n\cdots(n+k-2)[2(n+k-1)-1]a_{n+k-1} r^{n-1} \\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty} n\cdots(n+k-2)a_{n+k-1} r^{n-1} \\
&\leq \frac{1}{2\beta}\sum_{n=1}^{\infty} n\cdots(n+k-2)[2(n+k-1)+1]a_{n+k-2} r^{n-1} \\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty} n\cdots(n+k-2)a_{n+k-1} r^{n-1}
\end{aligned}$$

$$= \frac{1}{\beta}\sum_{n=1}^{\infty} n\cdots(n+k-2)\left[(n+k-1)+\frac{1}{2}\right]a_{n+k-2}r^{n-1}$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} n\cdots(n+k-2)a_{n+k-1}r^{n-1}$$

$$\leq \frac{1}{\beta}\sum_{n=1}^{\infty} n\cdots(n+k-2)na_{n+k-2}r^{n-1}$$

$$+ \left(k-\frac{1}{2}\right)\frac{1}{\beta}\sum_{n=1}^{\infty} n\cdots(n+k-2)a_{n+k-2}r^{n-1} + \frac{1}{2}\frac{f^{(k-1)}(r)}{r}$$

$$= \frac{1}{\beta}\sum_{n=2}^{\infty}(n-1)n\cdots(n+k-2)a_{n+k-2}r^{n-1}$$

$$+ \left(k+\frac{1}{2}\right)\frac{1}{\beta}\sum_{n=1}^{\infty} n\cdots(n+k-2)a_{n+k-2}r^{n-1} + \frac{1}{2}\frac{f^{(k-1)}(r)}{r}$$

$$= \frac{1}{\beta}\sum_{n=1}^{\infty} n\cdots(n+k-1)a_{n+k-1}r^{n} + \left(k+\frac{1}{2}\right)\frac{1}{\beta}f^{(k-1)}(r) + \frac{1}{2}\frac{f^{(k-1)}(r)}{r}$$

$$= \frac{r}{\beta}f^{(k)}(r) + \left(k+\frac{1}{2}\right)\frac{1}{\beta}f^{(k-1)}(r) + \frac{1}{2}f^{(k-1)}(r)r$$

$$\frac{\beta-r}{\beta}f^{(k)}(r) \leq \left(k+\frac{1}{2}\right)\frac{1}{\beta}f^{k-1}(r) + \frac{1}{2}\frac{f^{k-1}(r)}{r}.$$

Thus,

$$f^{(k)}(r) \leq \frac{1}{2}\frac{2k+1}{\beta-r}f^{(k-1)}(r) + \frac{1}{2}\frac{\beta}{r}\frac{1}{\beta-r}f^{(k-1)}(r)$$

$$= \frac{1}{2(\beta-r)}\left[2k+1+\frac{\beta}{r}\right]f^{(k-1)}(r).$$

Consequently,

$$f^{(k)}(r) \leq \frac{1}{2^k} \frac{(2k+1+\frac{\beta}{r})(2k-1+\frac{\beta}{r})\cdots(5+\frac{\beta}{r})(3+\frac{\beta}{r})}{(\beta-r)^k} f(r)$$

which completes the induction proof.

We now use inequality (1), Taylor's Theorem, and the binomial series to obtain for $0 < c \leq r < \beta$

$$\begin{aligned} f(r) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(r-c)^n \\ &\leq f(c) \sum_{n=0}^{\infty} \frac{(2n+1+\frac{\beta}{r})\cdots(5+\frac{\beta}{r})(3+\frac{\beta}{r})}{2^n n!(\beta-c)^n}(r-c)^n \\ &\leq f(c)\left[1 + \sum_{n=1}^{\infty} \frac{(2n+1+\frac{\beta}{r})\cdots(5+\frac{\beta}{r})(3+\frac{\beta}{r})}{2^n n!}\left(\frac{r-c}{\beta-c}\right)^n\right] \\ &= f(c)\left(\frac{\beta-c}{\beta-r}\right)^{\frac{3+\frac{\beta}{r}}{2}}. \end{aligned}$$

Next, let s be a number such that $0 \leq s < \beta$. We shall show that

$$f^{(n)}(s) \geq \frac{(2n+1)\cdots 5\cdot 3}{2^n(\delta-s)^n} f(s) \text{ for } n = 1, 2, \ldots .$$

Now,

$$\begin{aligned} f'(s) &= \sum_{n=1}^{\infty} n a_n s^{n-1} = \frac{1}{2}\sum_{n=1}^{\infty} 2n a_n s^{n-1} \\ &\geq \frac{1}{2}\sum_{n=1}^{\infty}(2n-1)a_n s^{n-1} \geq \frac{1}{2\delta}\sum_{n=1}^{\infty}(2n+1)a_{n-1}s^{n-1} \\ &= \frac{1}{2\delta}\sum_{n=0}^{\infty}(2n+3)a_n s^n = \frac{s}{\delta}f'(s) + \frac{3}{2\delta}f(s). \end{aligned}$$

Thus,

$$f^{'}(s) \geq \frac{3}{2(\delta - s)} f(s). \tag{2}$$

Assume the induction hypothesis that

$$\begin{aligned} f^{(k-1)}(s) &= \sum_{n=k-1}^{\infty} n(n-1)\cdots(n-k+2)a_n r^{n-k+1} \\ &\geq \frac{(2k-1)\cdots 5\cdot 3}{2^{k-1}(\delta - s)^{k-1}} f(s) \end{aligned}$$

for some integer $k \geq 2$. Then,

$$\begin{aligned} f^{(k)}(s) &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n s^{n-k} \\ &= \sum_{n=1}^{\infty}(n+k-1)(n+k-2)\cdots n a_{n+k-1} s^{n-1} \\ &= \frac{1}{2}\sum_{n=1}^{\infty}(2n+2k-2)(n+k-2)\cdots n a_{n+k-1} s^{n-1} \\ &\geq \frac{1}{2}\sum_{n=1}^{\infty} n\cdots(n+k-2)[2(n+k-1)-1]a_{n+k-1} s^{n-1} \\ &\geq \frac{1}{2\delta}\sum_{n=1}^{\infty} n\cdots(n+k-2)(2n+2k-1)a_{n+k-2} s^{n-1} \\ &= \frac{1}{2\delta}\sum_{n=0}^{\infty}(n+1)\cdots(n+k-1)(n+2k+1)a_{n+k-1} s^{n} \end{aligned}$$

$$= \frac{s}{\delta} \sum_{n=1}^{\infty} n(n+1)\cdots(n+k-1)a_{n+k-1}s^{n-1}$$

$$+ \frac{2k+1}{2\delta} \sum_{n=0}^{\infty} (n+k-1)\cdots(n+2)(n+1)a_{n+k-1}s^{n}$$

$$= \frac{s}{\delta} f^{(k)}(s) + \frac{2k+1}{2\delta} f^{(k-1)}(s).$$

Thus,

$$f^{(k)}(s) \geq \frac{1}{2} \frac{2k+1}{\delta - s} f^{k-1}(s)$$

and therefore (using the induction hypothesis)

$$f^{(k)}(s) \geq \frac{(2k+1)\cdots 5 \cdot 3}{2^k(\delta - s)^k} f(s)$$

which completes the induction proof.

Similarly we use inequality (2), Taylor's Theorem, and the binomial series to obtain for $0 \leq c \leq s < \beta < \delta$,

$$f(s) \geq f(c)\left[1 + \sum_{n=1}^{\infty} \frac{(2n+1)\cdots 5 \cdot 3}{2^n n!} \left(\frac{s-c}{\delta - c}\right)^n\right] = f(c)\left[\frac{\delta - c}{\delta - s}\right]^{\frac{3}{2}}.$$

Case 1. (β is achieved). There exists $m \geq 1$ such that

$$\beta = \frac{2m+1}{2m-1} r_m.$$

We shall show that $\beta = 1$. If $m = 1$ there is nothing to prove. Assume $m \geq 2$. Now $\beta > r_m$ and we can use $c = r_{m-1}$ and $r = r_m$ to get

$$\left[\frac{\beta - r_{m-1}}{\beta - r_m}\right]^{\frac{3+\frac{\beta}{r_m}}{2}} \geq \frac{f(r_m)}{f(r_{m-1})}.$$

But,

$$\frac{\beta}{r_m} = \frac{2m+1}{2m-1}.$$

Therefore,

$$\left[\frac{\beta - r_{m-1}}{\beta - r_m}\right]^{4m-1} \geq \left[\frac{m(2m+1)}{(m-1)(2m-1)}\right]^{2m-1}.$$

Now using the lemma we have

$$\left(\frac{m}{m-1}\right)^{2m-1} \geq \left(\frac{m+\frac{1}{2}}{m-\frac{1}{2}}\right)^{2m}.$$

Thus,

$$\left[\frac{\beta - r_{m-1}}{\beta - r_m}\right]^{4m-1} \geq \left[\frac{2m+1}{2m-1}\right]^{2m} \left[\frac{2m+1}{2m-1}\right]^{2m-1} = \left[\frac{2m+1}{2m-1}\right]^{4m-1}$$

and consequently,

$$\frac{\beta - r_{m-1}}{\beta - r_m} \geq \frac{2m+1}{2m-1}.$$

So

$$\beta - r_{m-1} \geq \frac{2m+1}{2m-1}\beta - \beta.$$

It follows that

$$\beta \geq \frac{2m-1}{2m-3}\beta.$$

Since $m-1 \geq 1$,

$$\beta \leq \frac{2m-1}{2m-3}r_{m-1}.$$

Therefore,

$$\beta = \frac{2m-1}{2m-3}r_{m-1}.$$

Proceeding as above it follows that $\beta = 1$.

Now by the definition of β we have

$$\frac{1}{r_n} \leq \frac{2n+1}{2n-1} \text{ for } n = 1, 2, \ldots .$$

But,

$$a_n = \frac{1}{r_1 r_2 \ldots r_n}$$

and hence, $a_n \leq 2n+1$. Thus,

$$3 = f\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{3}\right)^n \leq \sum_{n=0}^{\infty} (2n+1) \left(\frac{1}{3}\right)^n = 3$$

and so $a_n = 2n+1$. Consequently,

$$f(z) = \frac{1+z}{(1-z)^2} \text{ for all } |z| < 1$$

in this case.

Case 2. (β and δ are not achieved). Here for every $n = 1, 2, \ldots$

$$\beta < \frac{2n+1}{2n-1} r_n < \delta$$

Since β is an infimum, a subsequence of

$$\left\{ \frac{2n+1}{2n-1} r_n \right\}$$

must converge to β. But as the subsequence itself is convergent we must have

$$\lim_{n \to \infty} \frac{2n+1}{2n-1} r_n = \beta.$$

Then

$$\lim_{n \to \infty} r_n = \beta.$$

So $\beta = R$. Since δ is a supremum, we must also have

$$\lim_{n \to \infty} r_n = \delta.$$

Thus, $\beta = R = \delta$ and using the definitions of β and δ we have

$$\frac{2n+1}{2n-1} r_n = R$$

for all $n = 1, 2, \ldots$. That is, this case cannot occur.

Case 3. (β is not achieved but δ is). Here for every $n = 1, 2, \ldots$ we have

$$\beta < \frac{2n+1}{2n-1} r_n$$

and there is an integer $m \geq 1$ such that

$$\delta = \frac{2m+1}{2m-1} r_m.$$

If $m = 1$, there is nothing to prove. Assume $m \geq 2$. As in Case 2,

$$\lim_{n\to\infty} \frac{2n+1}{2n-1} r_n = \beta$$

so that $\beta = R$. In particular, $r_n < \beta < \delta$ for $n = 0, 1, \ldots$ and we can use $c = r_{m-1}$ and $s = r_m$ to get

$$\left(\frac{\delta - r_{m-1}}{\delta - r_m}\right)^3 \leq \left[\frac{m(2m+1)}{(m-1)(2m-1)}\right]^2.$$

But

$$\frac{m}{m-1} \leq \left(\frac{2m+1}{2m-1}\right)^2.$$

So

$$\frac{\delta - r_{m-1}}{\delta - r_m} \leq \frac{2m+1}{2m-1}$$

and the result follows as in Case 1.

4. Counterexamples. Here is a counterexample for problem 1.

Let

$$f(r) = \sum_{n=0}^{\infty} \frac{1}{e^n (n+1)^2}\, r^n$$

and let

$$g(r) = \sum_{n=0}^{\infty} \frac{e^n}{(n+1)^2} r^n.$$

Clearly,

$$r_n = e\left(1+\frac{1}{n}\right)^2 \text{ and } s_n = \frac{1}{e}\left(1+\frac{1}{n}\right)^2 \text{ for } n \geq 1.$$

Now

$$f(r_k) = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}\left[\left(1+\frac{1}{k}\right)^n\right]^2 = g(s_k) \text{ for } k \geq 1.$$

Moreover, since $r_0 = 0$ and $s_0 = 0$, it follows that $f(r_0) = 1 = g(s_0)$ and therefore $f(r_k) = g(s_k)$ for $k \geq 0$. However, $f \not\equiv g$.

Here is a counterexample for Problem 2.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function such that

(i) $a_n < 0$ for $n = 0, 1, 2, \ldots$

(ii) $a_1 = -1$ and $f(r_n) = e^n$ for $n = 0, 1, 2, \ldots$

and put $g(z) = -f(z)$. Then clearly $g(z)$ satisfies all the hypothesis of Theorem 2 and thus, $g(z) = e^z$ for all z. Consequently, $f(z) = -e^z$ for all z.

This shows that the condition $a_n > 0$ for $n = 0, 1, 2, \ldots$ cannot be relaxed to the condition $a_n \neq 0$ for $n = 0, 1, 2, \ldots$.

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