TANGENT SPACES OF MINKOWSKI SPACES

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Abstract. Let $\mathcal{K}_o$ be a class of strictly convex compact bodies in $\mathbb{R}^n$ which are centrally symmetric with respect to the origin o in its interior. Let $\overline{U}_i, \overline{U} \in \mathcal{K}_o$ for $i = 1, 2, \ldots$. In this paper, we prove that if $\partial\overline{U}_i = U_i$ converges to $\partial\overline{U} = U$ in the Hausdorff sense as i tends to infinity, then the tangent space $T_oM^n(U_i)$ of the Minkowski space $M^n(U_i)$ converges to the tangent space $T_oM^n(U)$ of $M^n(U)$.

1. Introduction and Preliminaries. To investigate Minkowski spaces we use the metric method and the geometry of geodesics. Direction spaces and tangent spaces are good tools for investigating the characteristics of these spaces (for Riemannian manifolds X the tangent space at a point of X is evidently the Euclidean tangent space of X). Tangent spaces of Minkowski spaces are especially good for providing information for Minkowski surface area [7]. Thus, it is very important and interesting to study them. Using the definition of the Minkowski angle of two geodesic directions we define the direction spaces and the tangent spaces of the Minkowski spaces and then we investigate some properties of Minkowski spaces influenced from the indicatrices of the spaces. The aim of this paper is to show that for centrisymmetric strictly convex compact bodies $\overline{U}_i$ and $\overline{U}$, if $U_i \underset{H}{\longrightarrow} U$ then

$$T_oM^n(U_i) \underset{H}{\longrightarrow} T_oM^n(U).$$

The n–dimensional Euclidean space E^n can be viewed as a space R^n, n–dimensional real space, with Euclidean norm $\|\cdot\|$. To generalize the situation we consider $\mathbb{R}^n$ as an n–dimensional normed space with an arbitrary norm $\|\cdot\|_m$. The norm $\|\cdot\|_m$ makes $\mathbb{R}^n$ a new metric space. This new space is usually called the n–dimensional Minkowski space M^n with metric m, defined by

$$m(x, y) = \|x - y\|_m.$$

We know that there is a centrisymmetric convex body $\overline{U}$ that satisfies the property

$$m(x, y) = \frac{2e(x, y)}{e(x', y')},$$

where x', y' are points on the intersection of the boundary $\partial\overline{U} = U$ of $\overline{U}$ with the line through the origin o which is parallel to the line passing through x and y.

Example 1. In E^2 let $\overline{U}$ be the convex hull of four points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. For $x \in E^2$ we define

$$\|x\|_m = \begin{cases} \frac{\|x\|}{\|\overline{x}\|}, & \text{if} \quad x \neq o \\ 0, & \text{if} \quad x = o, \end{cases}$$

where $\overline{x}$ is the point on the intersection of U with the ray passing through x emanating from o. Then $\|\cdot\|_m$ defines a Minkowski metric.

We call U the indicatrix of M^n and denote by $M^n(U)$ the n–dimensional Minkowski space with the indicatrix U. For details see [2, 3, 5, 6, 7]. Obviously the hypersurface U satisfies that $U = \{x \mid \|x\|_m = 1\}$. One can easily see that the boundary U of any centrally symmetric closed convex body $\overline{U}$ is the indicatrix of some Minkowski space. In this sense, we can use the term Minkowski space for the finite dimensional normed linear space. The metric m is invariant under all translations $x' = x + a$ for any constant $a \in M^n$. Of course there is a metric on $\mathbb{R}^n$ which is not Minkowski. From now on, to distinguish this from other metrics, we denote by e the standard Euclidean metric.

Example 2. Let E^n be the Euclidean space with metric e. Metrize $\mathbb{R}^n$ with a metric d defined as

$$d(x, y) = \begin{cases} e(x, y) + \frac{1}{2}, & \text{if} \quad x \neq y \\ 0, & \text{if} \quad x = y. \end{cases}$$

Then $\mathbb{R}^n$ with d is not Minkowski because for a scalar α we get

$$d(\alpha x, \alpha y) = |\alpha| e(x, y) + \frac{1}{2} \neq |\alpha| d(x, y),$$

so that d is not a norm on $\mathbb{R}^n$. However, the unit ball $B_d(x, 1)$ at x with the metric d is the ball centered at x of radius $\frac{1}{2}$ of the metric e, that is,

$$B_d(x, 1) = B_e\left(x, \frac{1}{2}\right).$$

So $B_d(x,1)$ is convex in the Euclidean sense. From this, we know that the Euclidean convexity of the unit ball does not guarantee the space to be Minkowski.

A geodesic in a Minkowski space M^n is a map $\sigma\colon I \to M^n$ such that for some real number $\alpha \geq 0$ and for every t on the interval $I \subset E^1$ there exists a neighborhood $V \subset I$ such that $m(\sigma(t_1), \sigma(t_2)) = \alpha e(t_1, t_2)$ for all $t_1, t_2 \in V$. If one can take $V = I$, then σ is said to be a minimizer. We know that a curve σ from a point x to a point y in the Minkowski space M^n is a minimizer from x to y if for any point z on the image of σ the relation

$$m(x,z) + m(z,y) = m(x,y)$$

holds. Now it is natural to define the convexity in M^n as follows. A set K in the Minkowski space is said to be convex if for any points $x, y \in K$ there is a minimizer from x to y contained entirely in K. Since a geodesic in the Euclidean space is also a geodesic in the Minkowski space, a convex subset of E^n is also a convex subset of M^n, but not conversely unless $\overline{U}$ is assumed to be strictly convex. Here the strict convexity means that for any points $x, y \in \overline{U}$ the interior points of the line segment $\overline{xy}$ from x to y are also interior points of $\overline{U}$. Now we have the following.

Theorem 1. Let $M^n(U)$ be a Minkowski space with non–strict convexity $\overline{U}$. Then there is a geodesic which is not a straight line in E^n.

Proof. Because $\overline{U}$ is not strictly convex we can find a line segment $\overline{xy}$ contained in U. We assert that $\overline{oy} \cup \overline{o(-x)}$ is a geodesic. Let $z \in \overline{oy}$. Then by definition we obtain

$$m(-x,z) + m(z,y) = \frac{e(-x,z)}{e(o,w)} + \frac{e(z,y)}{e(o,y)},$$

where w is the point of intersection of $\overline{(-x)(-y)}$ and the line through o and parallel to $\overline{(-x)z}$ in the hyperplane contains a triangle $\triangle(o,x,y)$. Because $\triangle ow(-y)$ is similar to $\triangle z(-x)(-y)$, we get

$$\frac{e(-x,z)}{e(o,w)} = \frac{e(-y,z)}{e(o,-y)}.$$

So

$$
\begin{aligned}
m(-x,z)+m(z,y) &= \frac{e(-y,z)}{e(o,-y)}+\frac{e(z,y)}{e(o,y)} \\
&= \frac{e(-y,z)+e(z,y)}{e(o,y)} \\
&= m(-x,y).
\end{aligned}
$$

This completes the proof.

2. Main Results. Now we investigate the tangent space of the n–dimensional Minkowski space M^n. The distortion of a map $f\colon X\to Y$ of metric spaces is defined as

$$\operatorname{dis}(f)=\sup_{x,y\in X}|d_X(x,y)-d_Y(f(x),f(y))|,$$

where d_X is a metric on X. An ε–mesh of a metric space X is a subset A such that for every $x\in X$ there is $a\in A$ with $d_X(a,x)<\varepsilon$. The uniform distance between metric spaces X,Y is defined as

$$u(X,Y)=\inf_f \operatorname{dis}(f),$$

where the infimum is taken over all bijections $f\colon X\to Y$. We say that X_i converges uniformly to X, denoted by

$$X_i \underset{u}{\longrightarrow} X,$$

if $u(X_i,X)\to 0$ as i tends to infinity. A sequence X_i of metric spaces is said to be Hausdorff convergent to a metric space X, denoted by

$$X_i \underset{H}{\longrightarrow} X,$$

if for every $\varepsilon>0$ there exists an ε–mesh X_ε in X which is the uniform limit of ε–meshes $(X_i)_\varepsilon$ in X_i.

Let $\mathcal{K}_o$ be a class of strictly convex compact bodies with nonempty interior in $\mathbb{R}^n$ which are centrally symmetric with respect to o in its interior. From now on we assume that $\overline{U}_i, \overline{U} \in \mathcal{K}_o$ for $i = 1, 2, \ldots$. Now we investigate properties of tangent spaces of Minkowski spaces. From the fact that a mesh of the hypersurface U determines a mesh of the Minkowski space $M^n(U)$, we have the following theorem.

Theorem 2. Let U_i and U be indicatrices of $M^n(U_i)$ and $M^n(U)$, respectively. If $U_i \underset{H}{\longrightarrow} U$, then $M^n(U_i) \underset{H}{\longrightarrow} M^n(U)$.

Proof. Let $\varepsilon > 0$. Let $r_i(x)$ be a radial function of indicatrix U_i defined as follows for x in the unit sphere S^{n-1} with center o,

$$r_i(x) = e(o, \overline{x}),$$

where $\overline{x}$ is the point on the intersection of U_i with the ray passing through x emanating from o. Let $r(x)$ be a radial function of the indicatrix U. Since

$$U_i \underset{H}{\longrightarrow} U,$$

we can choose m satisfying

$$0 < m \leq \min_{x \in S^{n-1}} \{r_i(x), r(x)\}$$

for $i \geq N$ for sufficiently large N. If we put $\delta = (m\varepsilon)/2$, then there is a δ–mesh U_δ in U which is the uniform limit of δ–meshes $(U_i)_\delta$ in U_i. Let

$$N_1 = \frac{\delta}{2} U_\delta = \left\{ \frac{\delta}{2} x \mid x \in U_\delta \right\}$$

and

$$N_k = \frac{k}{k-1} \alpha_{k-1} \left\{ z \in M^n(U) \mid m(x,z) + m(z,y) = m(x,y) \text{ for some } x, y \in N_{k-1} \right\}$$

for $k \geq 2$, where α_k is the greatest lower bound of the set

$$\{\|x\| \mid x \in N_k\}.$$

In a similar way, for each N_i we get $(N_i)_k$ for $k \geq 1$. Now let

$$(M^n(U))_\varepsilon = \bigcup_{k=1}^{\infty} N_k \quad \text{and} \quad (M^n(U_i))_\varepsilon = \bigcup_{k=1}^{\infty} (N_i)_k.$$

Since $\overline{U}$ and $\overline{U}_i$ are convex, $(M^n(U))_\varepsilon$ and $(M^n(U_i))_\varepsilon$ are ε–meshes of $M^n(U)$ and $M^n(U_i)$, respectively. Since $X_i \underset{u}{\longrightarrow} X$ if and only if $\alpha X_i \underset{u}{\longrightarrow} \alpha X$ for any scalar α,

$$(M^n(U_i))_\varepsilon \underset{u}{\longrightarrow} (M^n(U))_\varepsilon.$$

So

$$M^n(U_i) \underset{H}{\longrightarrow} M^n(U).$$

This completes the proof.

It should be noted that Theorem 2 says that the Euclidean space E^n is the Hausdorff limit of a sequence of Minkowski spaces since the Euclidean sphere is the Hausdorff limit of hypersurfaces which are the boundary of the centrisymmetric convex bodies.

From now on, we assume that U is differentiable and has positive finite curvature everywhere. Let σ_i and σ_j be two geodesic directions and $U(\sigma_i, \sigma_j)$ be the intersection of U and the hyperplane determined by σ_i, σ_j. Now we parametrize

$$\sqrt{\frac{\pi}{A(U(\sigma_i, \sigma_j))}}\, U(\sigma_i, \sigma_j),$$

where $A(U(\sigma_i, \sigma_j))$ is the area of $\overline{U}(\sigma_i, \sigma_j)$, the convex hull of $U(\sigma_i, \sigma_j)$ in the Minkowski plane

$$M^2\left(\sqrt{\frac{\pi}{A(U(\sigma_i, \sigma_j))}}\, U(\sigma_i, \sigma_j)\right),$$

by twice its sectorial area, ϕ, and write the equation of

$$\sqrt{\frac{\pi}{A(U(\sigma_i,\sigma_j))}}U(\sigma_i,\sigma_j)$$

as

$$\zeta = \zeta_{i,j}(\phi). \tag{1}$$

We define $\eta = \eta_{i,j}(\phi)$ by

$$\eta_{i,j}(\phi) = \frac{d\zeta_{i,j}(\phi)}{d\phi}.$$

We know that the isoperimetrix [1] $I(\sigma_i,\sigma_j)$ of the Minkowski plane

$$M^2\left(\sqrt{\frac{\pi}{A(U(\sigma_i,\sigma_j))}}U(\sigma_i,\sigma_j)\right)$$

is the curve with equation

$$\eta = \eta_{i,j}(\phi).$$

Then the function $\lambda = \lambda_{i,j}(\phi)$ defined by the equation

$$\frac{d\eta_{i,j}(\phi)}{d\phi} = -\lambda_{i,j}^{-1}(\phi)\zeta_{i,j}(\phi)$$

is called the Minkowski curvature of $I(\sigma_i,\sigma_j)$ at a point where the tangent has direction $\zeta_{i,j}(\phi)$ [4]. We define the Minkowski angle ω between the two directions $\sigma_i = \zeta(\phi_i)$ and $\sigma_j = \zeta(\phi_j)$, $\phi_i \le \phi_j$, by

$$\omega_{i,j} = \angle(\sigma_i,\sigma_j) = \int_{\phi_i}^{\phi_j} \lambda_{i,j}^{-1}(\phi)d\phi.$$

Definition 1. The direction space $D_x M^n$ at $x \in M^n$ is defined as the set of geodesics in M^n emanating from x with a metric d defined as

$$d(\sigma_i, \sigma_j) = \angle(\sigma_i, \sigma_j).$$

For three geodesics $\sigma_i, \sigma_j, \sigma_k$ let $\omega_{i,j}$ be the angle between two geodesics σ_i and σ_j. Then

$$\omega_{i,j} \leq \omega_{i,k} + \omega_{k,j}.$$

So the metric on $D_x M^n$ above is well–defined. Then we have the following.

Theorem 3. Let U_i and U be the indicatrices of $M^n(U_i)$ and $M^n(U)$, respectively. If $U_i \underset{H}{\longrightarrow} U$, then for any point $x \in U \cap (\bigcap_{i=1}^{\infty} U_i)$

$$D_x M^n(U_i) \underset{H}{\longrightarrow} D_x M^n(U).$$

Proof. Since the Minkowski metric is invariant under translations, it is sufficient to prove that

$$D_o M^n(U_i) \underset{H}{\longrightarrow} D_o M^n(U).$$

Let $\varepsilon > 0$. Let σ_i, σ_j be two geodesic directions. We denote the Minkowski and Euclidean element of arc of $I(\sigma_i, \sigma_j)$ by dL and dL_e, respectively. Put

$$m = \min_{i,j} \{ \| \zeta_{i,j}(\phi) \| \},$$

where $\zeta_{i,j}(\phi)$ is a parametrization of

$$\sqrt{\frac{\pi}{A(U(\sigma_i, \sigma_j))}} U(\sigma_i, \sigma_j)$$

as in (1). Then we can choose l_i, l_j so that

$$\omega_{i,j} = \int_{\phi_i}^{\phi_j} \lambda^{-1}(\phi) d\phi = \int_{l_i}^{l_j} dL.$$

Then we can put

$$\int_{l_i}^{l_j} dL = \int_{\overline{l_i}}^{\overline{l_j}} \frac{dL_e}{r},$$

where r is a radial function of

$$\sqrt{\frac{\pi}{A(U(\sigma_i, \sigma_j))}} U(\sigma_i, \sigma_j).$$

Then

$$\int_{\overline{l_i}}^{\overline{l_j}} \frac{dL_e}{r} \leq \frac{1}{m} \int_{\overline{l_i}}^{\overline{l_j}} dL_e. \tag{2}$$

Since

$$U_i \underset{H}{\longrightarrow} U,$$

there is a δ–mesh, $\delta = (m\varepsilon)/3$, U_δ in U which is the uniform limit of δ–meshes $(U_i)_\delta$ in U_i. For $x \in U_\delta \cup (U_i)_\delta$ let σ_x be the geodesic rays through x emanating from o. Put

$$(D_o M^n(U))_\varepsilon = \bigcup_{x \in U} [\sigma_x] \quad \text{and} \quad (D_o M^n(U_i))_\varepsilon = \bigcup_{x \in U_i} [\sigma_x].$$

From [2] $(D_o M^n(U))_\varepsilon$ and $(D_o M^n(U_i))_\varepsilon$ are ε–meshes of $D_o M^n(U)$ and $D_o M^n(U_i)$, respectively. Thus,

$$(D_o M^n(U_i))_\varepsilon \underset{u}{\longrightarrow} (D_o M^n(U))_\varepsilon.$$

So

$$D_o M^n(U_i) \underset{H}{\longrightarrow} D_o M^n(U).$$

This completes the proof.

Definition 2. The Euclidean cone $C(X) = X \times [0,\infty)/X \times 0$ over a metric space X has the metric defined as

$$d^2_{C(X)}(\overline{x},\overline{y}) = s^2 + t^2 - 2st\cos(\min\{d_X(x,y),\pi\}),$$

where $\overline{x} = (x,s),\ \overline{y} = (y,t)$.

We have the following example of the Euclidean cone which is not Minkowski.

Example 3. Let $S^n(r)$ be a sphere of radius $r \neq 1$ with length metric $d_{S^n(r)}$. Then the Euclidean cone $C(S^n(r))$ is not Minkowski.

The space $C(S^n(r))$ is isometric to $(\mathbb{R}^{n+1}, d_{\mathbb{R}^{n+1}})$ with a metric defined by

$$d^2_{\mathbb{R}^{n+1}}(x,y) = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos(d_{S^n(r)}(x',y')),$$

where a' is on the intersection $S^n(r) \cap \overrightarrow{oa}$ of $S^n(r)$ with the ray $\overrightarrow{oa}$ emanating from o and passing through a point a. The map $i\colon C(S^n) \to \mathbb{R}^{n+1}$ defined by

$$i(x,t) = tx$$

is an isometry from $C(S^n)$ to $\mathbb{R}^{n+1}$. But

$$d_{\mathbb{R}^{n+1}}(i(x,t), i(y,s)) = d_{\mathbb{R}^{n+1}}(tx, sy)$$

$$= \sqrt{s^2 + t^2 - 2st\cos(d_{S^n(r)}(x,y))} = d_{C(S^n(r))}((x,t),(y,s)),$$

since the angle between the rays $\overrightarrow{o(tx)}$ and $\overrightarrow{o(sy)}$ is $d_{S^n(r)}(x,y)$. We have that $C(S^n(r))$ is not Minkowski, since $d_{C(S^n(r))}(x+a, y+a) \neq d_{C(S^n(r))}(x,y)$ in general.

Definition 3. The tangent space T_xX at a point $x \in X$ is defined as the Euclidean cone $C(D_xX)$ over the direction space D_xX at x of X.

Now we have a theorem concerning the convergence of tangent spaces of Minkowski spaces.

Theorem 4. Let U_i and U be the indicatrices of $M^n(U_i)$ and $M^n(U)$, respectively. If $U_i \underset{H}{\longrightarrow} U$, then for any point $x \in U \cap (\bigcap_{i=1}^{\infty} U_i)$

$$T_xM^n(U_i) \underset{H}{\longrightarrow} T_xM^n(U).$$

Proof. Since the Minkowski metric is invariant under translations it is sufficient to prove that

$$T_oM^n(U_i) \underset{H}{\longrightarrow} T_oM^n(U).$$

Let $\varepsilon > 0$. Choose a δ–mesh, $\delta = (m\varepsilon)/6$, where $m = \min_{i,j}\{\|\zeta_{i,j}(\phi)\|\}$ as in the proof of Theorem 3 and U_δ in U, the uniform limit of δ–meshes $(U_i)_\delta$ in U_i. For $x \in U_\delta \cup (U_i)_\delta$ let σ_x be the geodesic rays through x emanating from o. Let $A_1 = \bigcup_{x\in U}[\sigma_x]$ and A_k is the set of elements $[\sigma_y] \in D_oM^n(U)$ such that $y \in U$ and $\angle(\sigma_x, \sigma_y) + \angle(\sigma_y, \sigma_z) = \angle(\sigma_x, \sigma_z)$ for some $[\sigma_x], [\sigma_z] \in A_{k-1}$ for $k \geq 2$. Similarly, we obtain $(A_i)_k$ for $k \geq 1$. Now we put

$$(T_oM^n(U))_\varepsilon = \bigcup_{k=1}^{\infty} A_k \times \frac{k\varepsilon}{2} \quad \text{and} \quad (T_oM^n(U_i))_\varepsilon = \bigcup_{k=1}^{\infty} (A_i)_k \times \frac{k\varepsilon}{2}.$$

Without loss of generality, consider $0 < \varepsilon < 1$ and sufficiently small. Then for $\overline{x} = (x, t) \in T_oM^n(U)$ and $(k\varepsilon)/2 \leq t \leq ((k+1)\varepsilon)/2$, choose a point $\overline{x}' = (x, (k\varepsilon)/2)$ in $T_oM^n(U)$. Then choose a point $\overline{x}'_{\frac{\varepsilon}{2}}$ in A_k such that $d(\overline{x}', \overline{x}'_{\frac{\varepsilon}{2}}) < \frac{\varepsilon}{2}$. Then

$$d(\overline{x}, \overline{x}'_{\frac{\varepsilon}{2}}) \leq d(\overline{x}, \overline{x}') + d(\overline{x}', \overline{x}'_{\frac{\varepsilon}{2}}) < \varepsilon.$$

Thus, $(T_oM^n(U))_\varepsilon$ is an ε–mesh of $T_oM^n(U)$ and similarly $(T_oM^n(U_i))_\varepsilon$ is an ε–mesh of $T_oM^n(U_i)$. Since $D_oM^n(U)$ and $D_oM^n(U_i)$ are strictly interior, that is, every two points have a midpoint, we can choose A_k and $(A_i)_k$ so that

$$(T_oM^n(U_i))_\varepsilon \underset{u}{\longrightarrow} (T_oM^n(U))_\varepsilon.$$

So we have

$$T_oM^n(U_i) \underset{H}{\longrightarrow} T_oM^n(U).$$

This completes the proof.

Acknowledgement. The authors thank the referee for his helpful suggestions and comments. This work was supported by the Brain Korea 21 Project from the Korean Ministry of Education and KOSEF 2000–2–10200–001–3.

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