AN ALGEBRAIC REMARK ON THE FOURIER SERIES OF A TRIGONOMETRIC POLYNOMIAL

Daniel Cohen and Thierry Dana–Picard

Abstract. In a previous note [1], we showed how to linearize trigonometric polynomials using complex numbers, Euler's formula, and DeMoivre's formula. Here we explain why this gives, in fact, the Fourier series of these trigonometric polynomials.

Let V be a prehilbertian vector space, i.e. a vector space (possibly of infinite dimension) together with an inner product, denoted $\langle \cdot, \cdot \rangle$. Now let W be a vector subspace of V, with an orthonormal basis $\overrightarrow{w_1}, \ldots, \overrightarrow{w_n}, \ldots$. If W is of finite dimension, the orthogonal projection of a vector $\overrightarrow{v} \in V$ onto W is the vector

$$P_W(\overrightarrow{v}) = \sum_n \langle \overrightarrow{w_n}, \overrightarrow{v} \rangle \overrightarrow{w_n}. \tag{1}$$

If the dimension of W is infinite, the formal infinite sum in (1) will still be called the orthogonal projection of $\overrightarrow{v}$ onto W.

Let V be the vector space of functions defined on $\mathbb{R}$ and periodic with a period dividing 2π. On V we define an inner product by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx.$$

This is the inner product defining the L_2–norm in our functional space.

Now consider the subspace W generated by the constant function 1 and the functions c_n and s_n such that for any $x \in \mathbb{R}$, $c_n(x) = \cos nx$ and $s_n(x) = \sin nx$, for all $n \in \mathbb{N}$.

For a function $f \in V$, its orthogonal projection F onto the subspace W is determined by the formula

$$F = \langle 1, f \rangle \cdot 1 + \sum_{n \geq 0} a_n c_n + b_n s_n$$

where $a_n = \langle c_n, f\rangle = \int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \langle s_n, f\rangle = \int_{-\pi}^{\pi} f(x)\sin nx\,dx$. The numbers a_n and b_n are the so–called *Fourier coefficients* of f and the series on the left is the Fourier series of f.

In [1] we described a method of linearization of trigonometric polynomials based on the usage of complex numbers, DeMoivre's formula, and Newton binomial development. For the reader's sake, we recall the method with an example. Let $f(x) = \sin^2 x \cos^3 x$. By Euler's formula, $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$. Substituting for $\cos x$ and $\sin x$ in $f(x)$, we obtain

$$
\begin{aligned}
f(x) &= \left(\frac{1}{2i}(e^{ix} - e^{-ix})\right)^2 \cdot \left(\frac{1}{2}(e^{ix} + e^{-ix})\right)^3 \\
&= -\frac{1}{32}(e^{ix} - e^{-ix})^2(e^{ix} + e^{-ix})^3 \\
&= -\frac{1}{32}(e^{5ix} + e^{-5ix} + e^{3ix} + e^{-3ix} - 2(e^{ix} + e^{-ix})) \\
&= -\frac{1}{32}(2\cos 5x + 2\cos 3x - 4\cos x) \\
&= -\frac{1}{16}\cos 5x - \frac{1}{16}\cos 3x - \frac{1}{8}\cos x.
\end{aligned}
$$

Proposition 1. Let $p, q \in \mathbb{N}$. If $f(x) = \cos^p x \cdot \sin^q x$, then the linearization afforded by the method in [1] is actually the Fourier series of the function f.

Of course the same result holds for any (finite) linear combination of terms of the form $\cos^p x \cdot \sin^q x$.

Proof. The linearization method from [1] provides a decomposition of f as a linear combination of sines and cosines, i.e. a linear decomposition with respect to the above mentioned basis $\{1, c_1, s_1, c_2, s_2, \dots\}$, proving by the way that any trigonometric polynomial function f (i.e. $f(x)$ is a finite linear combination of terms of the form $\cos^p x \cdot \sin^q x$) belongs to W. The Fourier series of the function f is a decomposition on the same basis of the orthogonal projection of f onto W. As $f \in W$, these two decompositions are identical.

Reference

1. T. Dana–Picard and D. Cohen, "Linearization of Trigonometric Polynomials and Integrals," *Missouri Journal of Mathematical Sciences,* 11 (1999), 87–92.

Daniel Cohen
Department of Applied Mathematics
Jerusalem College of Technology
Havaad Haleumi Str. 21
POB 16031
Jerusalem 91160, Israel

Thierry Dana–Picard
Department of Applied Mathematics
Jerusalem College of Technology–Machon Lev
21 Havaad Haleumi St.
POB 16031
Jerusalem 91160, Israel
email: dana@math.jct.ac.il