SOME RESULTS ON n-STABLE RINGS

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Abstract. All rings are commutative rings with identity, and $J(R)$ denotes the Jacobson radical of the ring R. For any fixed integer $n \geq 1$, it is shown that the class of all n-stable rings is properly contained in the class of all $n+1$-stable rings. Results are given showing the connection between several types of rings whose finite sequences satisfy different stability conditions, some involving $J(R)$. It is shown that in the strongly n-stable case, it suffices to check whether the $n+1$-tuples satisfy the stable condition. In addition to other results and an example of a ring which is not n-stable for any integer $n \geq 1$, examples are given to show the distinction between the different types of stability cases. Finally, in the last section, some surjective mapping properties of a generalized form of $GL_n(R)$ and $SL_n(R)$ in connection to some stable conditions in the ring R are investigated.

1. Preliminaries. All rings are commutative rings with identity and for any ring R, $J(R)$ denotes the Jacobson radical of R. R is called a semisimple ring whenever $J(R) = (0)$. For any integer $s \geq 1$, a sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements of R is said to be stable provided that the ideal $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1+b_1a_{s+1}, \ldots, a_s+b_sa_{s+1})$ for some $b_1, b_2, \ldots, b_s \in R$ and also $a_1, a_2, \ldots, a_s, a_{s+1}$ is called a unimodular sequence whenever $(a_1, a_2, \ldots, a_s, a_{s+1}) = R$. For a fixed integer $n \geq 1$, we shall say R is n-stable, stable for the case $n = 1$, whenever for all $s \geq n$, any unimodular sequence of size $s+1$ is stable. It is clear that any n-stable ring is also m-stable for any integer $m \geq n$. R is a B-ring if for any unimodular sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ with $s \geq 2$ and $(a_1, a_2, \ldots, a_{s-1}) \not\subseteq J(R)$, there exists $b \in R$ such that $1 \in (a_1, a_2, \ldots, a_s + ba_{s+1})$. R is a strongly B-ring (SB-ring) provided that for any $d, a_1, a_2, \ldots, a_s, a_{s+1} \in R$ with $s \geq 2$, $(a_1, a_2, \ldots a_{s-1}) \not\subseteq J(R)$ and $d \in (a_1, a_2, \ldots, a_s, a_{s+1})$, then there exists $b \in R$ such that $d \in (a_1, a_2, \ldots, a_s + ba_{s+1})$. For general information on these subjects, the reader is referred to [1, 5, 6]. Note that the statement "R is n-stable" is an epitomized version of "n is in the stable range of R" which Estes and Ohm termed in [1].

Now for the sake of reference, we state the following proposition which is Theorem 3.4 in [2].

Proposition 1.1. Any n-dimensional commutative integral domain is $n+1$-stable and if R is an arbitrary n-dimensional commutative ring, then it is $n+2$-stable.

Remark. Theorem 2.3 in [1], which we state here, provides another type of dimensional criterion for the stable range in commutative rings. Let, for each ideal A of R, $J(A)$ denote the intersection of all maximal ideals of R containing A, and $J = \{\text{ideals } A \text{ of } R \mid J(A) = A\}$. As usual we denote the Krull dimension of R by $dimR$ and by $dim_J R$, we mean the sup of the length of the chains of prime ideals in J. Thus, $dim_J R \leq dim\, R$. R is called J-Noetherian provided the ideals of J satisfy the ascending chain condition. Theorem 2.3 in [1] states that R is $n+1$-stable whenever R is J-Noetherian and $dim_J R \leq n$. Following the notations of [1], let, for any ring R and positive integers $s \leq t$, $M(R, s \times t)$, be the set of all $s \times t$ matrices over R; $GL(R, s \times t)$ be the set of all $s \times t$ matrices whose $s \times s$ subdeterminants of each generate the unit ideal; and $SL(R, s \times s)$ denote the set of all $s \times s$ matrices of determinant 1. Further, the condition s^* is defined as follows: for every $\alpha \in GL(R, 1 \times s)$ there exists $M \in M(R, s-1 \times s)$ such that $\alpha \times M \in SL(R, s \times s)$. Here, $\alpha \times M$ is a matrix with the first row α and its bottom $s-1$ rows equal to those of M.

Let K be a field and $X_1, X_2, \dots, X_n$ be n indeterminates over K. According to an equivalent of Serres Theorem in [3], $K[X_1, X_2, \dots, X_n]$ satisfies the s^* condition for all $s \geq 2$. For K the field of real numbers, Estes and Ohm in [1] proved that if $p_n = X_1^2 + X_2^2 + \cdots + X_n^2 - 1$ and $(X_1, X_2, \dots, X_n, p_n)$ is stable in $K[X_1, X_2, \dots, X_n]$, then $K[X_1, X_2, \dots, X_n]$ does not satisfy m^* for all $m \geq n \geq 2$. From this and the validity of Serres Theorem in [4], we can extend Corollary 8.1 in [1] as follows.

Proposition 1.2. Assume K is the field of real numbers and $n \geq 2$. Then $(X_1, X_2, \dots, X_n, X_1^2 + X_2^2 + \cdots + X_n^2 - 1)$ is not stable in $K[X_1, X_2, \dots, X_n]$.

For the construction of the next example the following results are required.

Remark. Later in this paper it is shown that the n-stable property is preserved under the ring homomorphism. In [9], it is shown that $R[X]$ can never be stable for any ring R. Also in [5], it is proved that $R[X]$ is a SB-ring, consequently, a member of the class of 2- stable rings, if and only if R is a field.

Example 1.1. For each $n \geq 1$ let $\mathcal{S}_n$ denote the class of all n-stable rings. It is clear that $\mathcal{S}_1$ is nonempty since any field is stable. Now, by virtue of the above remark together with Propositions 1.1 and 1.2, it is clear that $\mathcal{S}_n$ is properly contained in $\mathcal{S}_{n+1}$ for any $n \geq 1$. Thus, $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3 \subset \cdots$ is an infinite proper chain. Moreover, let $R_1 = K$ be the field of real numbers and $R_n = K[X_1, X_2, \dots, X_{n-1}]$ for $n \geq 2$. Hence, $\prod_{i \geq 1} R_i$ the direct product of these rings is not n-stable for any $n \geq 1$ since the homomorphic image of an n-stable ring is again n-stable.

Remark. Since every Noetherian ring is J-Noetherian, Theorem 2.3 in [1] can also be applied in the argument of the above example instead of Proposition 1.1.

Next, the concept of n-stable rings, B-rings, and SB-rings in a natural way is generalized and investigated.

Definition 1.1. For any fixed integers $n \geq 1$ and $1 \leq k \leq n$, a sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements of R with $s \geq n$ is said to be (n,k)-stable if $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1 + b_1 a_{s+1}, a_2 + b_2 a_{s+1}, \ldots, a_k + b_k a_{s+1}, a_{k+1}, \ldots, a_s)$ for some $b_1, b_2, \ldots, b_k \in R$. R is an (n,k)-stable ring if any unimodular sequence of size larger than n is (n,k)-stable, and it is called a strongly (n,k)-stable ring whenever any sequence of size larger than n is (n,k)-stable. R is a strongly n-stable ring if any sequence of size larger than n is stable.

Definition 1.2. For fixed integers $n \geq 2$ and $1 \leq k \leq n$, R is an $(n,k)_J$-stable ring if for all $s \geq n$ any unimodular sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ with $(a_1, a_2, \ldots, a_{s-1}) \nsubseteq J(R)$ is (n,k)-stable. R is a strongly $(n,k)_J$-stable ring if for all $s \geq n$ and any $d, a_1, a_2, \ldots, a_s, a_{s+1} \in R$ with $(a_1, a_2, \ldots, a_{s-1}) \nsubseteq J(R)$ and $d \in (a_1, a_2, \ldots, a_s, a_{s+1})$, then there exist $b_1, b_2, \ldots, b_k \in R$ such that $d \in (a_1 + b_1 a_{s+1}, \ldots, a_k + b_k a_{s+1}, a_{k+1}, \ldots, a_s)$. For any fixed integer $n \geq 2$, R is said to be n_J-stable whenever for all $s \geq n$, any unimodular sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ with $(a_1, a_2, \ldots, a_{s-1}) \nsubseteq J(R)$ is stable, and it is called strongly n_J-stable if for all $s \geq n$ and any $d, a_1, a_2, \ldots, a_s, a_{s+1} \in R$ with $(a_1, a_2, \ldots, a_{s-1}) \nsubseteq J(R)$ and $d \in (a_1, a_2, \ldots, a_s, a_{s+1})$, then there exist $b_1, b_2, \ldots, b_s \in R$ such that $d \in (a_1 + b_1 a_{s+1}, \ldots, a_s + b_s a_{s+1})$.

Definition 1.3. For any fixed integers $n \geq 2$ and $1 \leq k \leq n$, a sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements of R with $s \geq n$, is said to be $(n, \bar{k})$-stable provided that the ideal $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1, a_2, \ldots, a_{s-k}, a_{s-(k-1)} + b_k a_{s+1}, \ldots, a_s + b_1 a_{s+1})$ for some $b_1, b_2, \ldots, b_k \in R$. A ring R is said to be $(n, \bar{k})_J$-stable if for all $s \geq n$, any unimodular sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ with $(a_1, a_2, \ldots, a_{s-1}) \nsubseteq J(R)$ is $(n, \bar{k})$-stable. For all $s \geq n$ and any $d, a_1, a_2, \ldots, a_s, a_{s+1} \in R$ with $(a_1, a_2, \ldots, a_{s-1}) \nsubseteq J(R)$, R is said to be a strongly $(n, \bar{k})_J$-stable ring provided that $d \in (a_1, a_2, \ldots, a_s, a_{s+1})$ implies $d \in (a_1, a_2, \ldots, a_{s-k}, a_{s-(k-1)} + b_k a_{s+1}, \ldots, a_s + b_1 a_{s+1})$ for some $b_1, b_2, \ldots, b_k \in R$.

Remark. It is not difficult to show that any finite sequence of elements of R with a unit term is always stable in R.

Lemma 1.1. In a ring R, any unimodular sequence $a_1, a_2, \ldots, a_s, a_{s+1} \in R$ is stable if $\{ a_1, a_2, \ldots, a_s, a_{s+1} \} \cap J(R) \neq \emptyset$. More precisely, if $a_i \in J(R)$ for some $1 \leq i \leq s$, then $(a_1, \ldots, a_{i-1}, a_i + a_{s+1}, a_{i+1}, \ldots, a_s) = R$, and for the case $i = s+1$, $(a_1 + a_{s+1}, a_2, \ldots, a_s) = (a_1, a_2 + a_{s+1}, \ldots, a_s) = \cdots = (a_1, a_2, \ldots, a_s + a_{s+1}) = R$.

Proof. Without loss of generality, assume $a_1 \in J(R)$. Now if $(a_1 + a_{s+1}, a_2, \ldots, a_s) \neq R$, then there exists a maximal ideal M of R such that $(a_1 + a_{s+1}, a_2, \ldots, a_s) \subseteq M$ which implies $R \subseteq M$. This is a contradiction.

The following lemma provides some equivalent criteria for the definitions of strongly n_J-stable, strongly $(n,k)_J$-stable, and strongly $(n,\bar{k})_J$- stable rings.

Lemma 1.2. For fixed integers $n \geq 2$ and $1 \leq k \leq n$, the following results are true.

i) R is strongly n_J-stable if and only if any sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ with $s \geq n$ and $(a_1, a_2, \ldots, a_{s-1}) \not\subseteq J(R)$ is stable.
ii) R is strongly $(n,k)_J$-stable if and only if any sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ with $s \geq n$ and $(a_1, a_2, \ldots, a_{s-1}) \not\subseteq J(R)$ is (n,k)-stable.
iii) R is strongly $(n,\bar{k})_J$-stable if and only if any sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ with $s \geq n$ and $(a_1, a_2, \ldots, a_{s-1}) \not\subseteq J(R)$ is $(n,\bar{k})$-stable.

Proof. We just make an argument for the first part and leave the other parts to the reader. Suppose $a_1, a_2, \ldots, a_s, a_{s+1}$ is a sequence in R with $(a_1, a_2, \ldots, a_{s-1}) \not\subseteq J(R)$. By the definition, $a_{s+1} \in (a_1, a_2, \ldots, a_s, a_{s+1})$ implies $a_{s+1} \in (a_1 + b_1 a_{s+1}, \ldots, a_s + b_s a_{s+1})$ for appropriate $b_1, b_2, \ldots, b_s \in R$, and this forces $(a_1, a_2, \ldots, a_s, a_{s+1}) \subseteq (a_1 + b_1 a_{s+1}, \ldots, a_s + b_s a_{s+1})$.

2. Some Basic Algebraic Properties.

Theorem 2.1.

i) For any fixed integer $n \geq 2$, R is n-stable if and only if it is n_J-stable.
ii) For fixed integers $n \geq 2$ and $1 \leq k \leq n$, R is an (n,k)-stable ring if and only if it is $(n,k)_J$-stable.
iii) For $n \geq 2$ and $1 \leq k \leq n$ if R is an $(n,k)_J$-stable ring, then it is $(n,\bar{k})_J$-stable, and the converse of the statement is also true for $2 \leq k \leq n$.
iv) For $n \geq 2$ if R is a stable $(n,\bar{1})_J$- stable ring, then it is an $(n,1)_J$-stable ring.
v) For $n \geq 4$, $2 \leq k \leq n$, and $k \neq n-1$, R is strongly $(n,k)_J$-stable if and only if it is strongly $(n,\bar{k})_J$-stable. For $k = 1$, the necessary part is also true.

Proof. Apply Lemma 1.1 for the first three parts. The other parts follow directly from the definition.

Remark. In the next section, it is shown that the class of all strongly 2-stable rings is properly contained in the class of all strongly 2_J-stable rings. Furthermore, it is also shown that the class of all strongly 2_J- stable rings is properly contained in the class of all 2-stable rings.

Theorem 2.2.

i) For fixed integers $n \geq 1$ and $1 \leq k \leq n$, R is strongly (n,k)-stable (respectively, (n,k)-stable) if and only if all (respectively, unimodular) sequences of size $n+1$ are (n,k)-stable.

ii) For any fixed integer $n \geq 1$, R is strongly n-stable (respectively, n-stable) if and only if any (respectively, unimodular) sequence of size $n+1$ is stable.

iii) For fixed integers $n \geq 2$ and $1 \leq k \leq n$, R is $(n,k)_J$-stable if and only if any unimodular sequence $a_1, a_2, \ldots, a_n, a_{n+1}$ with $(a_1, a_2, \ldots, a_{n-1}) \not\subseteq J(R)$ is (n,k)-stable.

iv) For $n \geq 3$ and $1 \leq k \leq n$ with $k \neq n-1$, R is strongly $(n,k)_J$-stable if and only if any sequence $a_1, a_2, \ldots, a_n, a_{n+1}$ with $(a_1, a_2, \ldots, a_{n-1}) \not\subseteq J(R)$ is (n,k)-stable.

v) For $n \geq 2$, R is strongly n_J-stable (respectively, n_J-stable) if and only if all (respectively, unimodular) sequences $a_1, a_2, \ldots, a_n, a_{n+1}$ with $(a_1, a_2, \ldots, a_{n-1}) \not\subseteq J(R)$ are stable.

Proof. A proof by induction is given for part (iv), and the other parts are left to the reader. Note that for the proof of unimodular cases, replace 1 with a_{n+2} in the following argument and also apply Lemma 1.1 whenever $J(R)$ is involved. Assume $a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}$ is a sequence in R with $(a_1, a_2, \ldots, a_n) \not\subseteq J(R)$. Thus, $a_{n+2} \in (a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2})$ implies $a_{n+2} = \sum_{i=1}^{n+2} a_i x_i = \sum_{i=1}^{n} a_i x_i + l$ for some $x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2} \in R$ and $l = a_{n+1}x_{n+1} + a_{n+2}x_{n+2}$. Now $a_{n+2} \in (a_1, a_2, \ldots, a_{n-1}, a_n, l)$ and either $(a_1, a_2, \ldots, a_{n-1}) \not\subseteq J(R)$ or $(a_1, a_2, \ldots, a_{n-1}) \subseteq J(R)$. Here we continue the argument only for the case $(a_1, a_2, \ldots, a_{n-1}) \subseteq J(R)$ and leave the other case to the reader. Thus, in this case, $(a_1, a_2, \ldots, a_{n-2}, a_n) \not\subseteq J(R)$ and for appropriate $b_1, b_2, \ldots, b_k \in R$, $a_{n+2} \in (a_1, a_2, \ldots, a_{n-1}, a_n, l) = (a_1, a_2, \ldots, a_{n-2}, a_n, a_{n-1}, l)$ which implies that $a_{n+2} \in (a_1 + b_1 l, a_2 + b_2 l, \ldots, a_k + b_k l, \ldots, a_{n-1}, a_n) \subseteq (a_1 + b_1 x_{n+2} a_{n+2}, a_2 + b_2 x_{n+2} a_{n+2}, \ldots, a_k + b_k x_{n+2} a_{n+2}, \ldots, a_n, a_{n-1}, a_{n+1})$.

Remark. The argument in the proof of Theorem 1.2 in [8] on B-rings can also be applied as a non-inductive direct approach for the proof of the above theorem.

From the above results, it is easy to see that R is strongly (n,n)-stable (respectively, (n,n)-stable) if and only if it is strongly n-stable (respectively, n-stable) for all $n \geq 1$. Next, we give a direct proof of this fact in the following theorem.

Theorem 2.3. For any fixed integer $n \geq 1$, R is strongly (n,n)-stable (respectively, (n,n)-stable) if and only if it is strongly n-stable (respectively, n-stable).

Proof. Assume R is strongly n-stable and $(a_1, a_2, \ldots, a_s, a_{s+1})$ with $s \geq n$ is an ideal of R. Thus, $a_{s+1} = \sum_{i=1}^{s+1} a_i x_i = \sum_{i=1}^{n} a_i x_i + l$ for some $x_1, x_2, \ldots, x_s, x_{s+1} \in R$

and $l = a_{n+1}x_{n+1} + \cdots + a_s x_s + a_{s+1}x_{s+1}$. Hence, for appropriate $b_1, b_2, \ldots, b_n \in R$, we have $a_{s+1} \in (a_1 + b_1 l, \ldots, a_n + b_n l) \subseteq (a_1 + b_1 x_{s+1} a_{s+1}, \ldots, a_n + b_n x_{s+1} a_{s+1}, a_{n+1} + 0a_{s+1}, \ldots, a_s + 0a_{s+1})$ which easily implies the result.

Remark. From the above result and Theorem 2.1, it is clear that for all $n \geq 2$, R is $(n, n)_J$-stable if and only if it is n_J-stable.

For the sake of reference, regardless of all possible equivalent stability cases, the following theorem will be stated for all rings that are defined in Definitions 1.1 through 1.3 above.

Theorem 2.4. Let $A \subseteq J(R)$ be an ideal of R.

i) For fixed integers $n \geq 1$ and $1 \leq k \leq n$, R is (n, k)-stable (respectively, n-stable) if and only if R/A is (n, k)-stable (respectively, n-stable). Further, the necessary part is always true for any ideal A of R.
ii) For $n \geq 1$ and $1 \leq k \leq n$, the homomorphic image of a strongly (n, k)-stable (respectively, strongly n-stable) ring is again a strongly (n, k)-stable (respectively, strongly n-stable) ring.
iii) For $n \geq 2$ and $1 \leq k \leq n$, R is $(n, k)_J$-stable (respectively, $(n, \bar{k})_J$-stable, n_J-stable) if and only if R/A is $(n, k)_J$-stable (respectively, $(n, \bar{k})_J$-stable, n_J-stable). The necessary part is always true for any ideal A of R.
iv) Let $n \geq 2$ and $1 \leq k \leq n$ be fixed integers, then the homomorphic image of a strongly $(n, k)_J$- stable (respectively, strongly $(n, \bar{k}_J$-stable, strongly n_J-stable) ring is a strongly $(n, k)_J$-stable (respectively, strongly $(n, \bar{k})_J$-stable, strongly n_J-stable).
v) Let $\{ R_i \mid i \in I \}$ be a family of rings. For fixed integers $n \geq 1$ and $1 \leq k \leq n$, the direct product $\prod_{i \in I} R_i$ is strongly (n, k)-stable (respectively, (n, k)-stable, n-stable, strongly n-stable) if and only if R_i is strongly (n, k)-stable (respectively, (n, k)-stable, n-stable, strongly n-stable) for each $i \in I$. Also, for $n \geq 2$ and $1 \leq k \leq n$, $\prod_{i \in I} R_i$ is $(n, k)_J$-stable (respectively, $(n, \bar{k})_J$-stable, n_J-stable) if and only if each factor of the product is $(n, k)_J$-stable (respectively, $(n, \bar{k})_J$-stable, n_J-stable).
vi) Let $\{ R_i \mid i \in I \}$ be a family of semisimple rings. For fixed integers $n \geq 2$ and $1 \leq k \leq n$, the direct product $\prod_{i \in I} R_i$ is strongly $(n, k)_J$-stable (respectively, strongly n_J-stable) if and only if each factor of the product is a strongly $(n, k)_J$-stable (respectively, strongly n_J-stable), and also the result holds for strongly $(n, \bar{k})_J$-stable rings whenever $2 \leq k \leq n$.

Proof. Follow the definitions, use Lemmas 1.1 and 1.2, and apply the technique which is given in the Proof of Theorem 2.2 above.

Remark. In the next section, it is shown that the product of two strongly 2_J-stable rings is not always a strongly 2_J-stable ring.

3. Some Examples and Applications. Besides some other results in [9], it is shown that $R[X]$ can never be stable and Artinian rings are always stable. In [7], it is proved that a formal power series with any number of indeterminates over a ring R is n-stable if and only if R is n-stable. See [8] for some improved results on B-rings, and also see Example 1.1 above. Finally in [10], as an application of SB-rings, it is shown that $R[X]$ can never be a Prüfer domain whenever R is a non-field Noetherian integral domain.

Next, we study the stability conditions of $Z_m[X]$. A ring R is completely primary if each element of R is either a unit or a nilpotent. By Theorem 2.7 (respectively, Theorem 3.4) in [5], $R[X]$ is a B-ring (respectively, SB-ring) if and only if R is a completely primary ring (respectively, a field). Note that every B-ring is 2-stable and every SB-ring is strongly 2_J-stable. Now from this and Theorem 2.4 above, we state the following example.

Example 3.1. For any integer $m = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$ with $p_1, p_2, \ldots, p_k$ distinct primes and each of $t_1, t_2, \ldots, t_k$ a positive integer,

$$Z_m[X] = Z_{p_1^{t_1}}[X] \times Z_{p_2^{t_2}}[X] \times \cdots \times Z_{p_k^{t_k}}[X]$$

is

i) a 2-stable ring which is not a B-ring whenever $k \geq 2$, or
ii) a B-ring which is not a SB-ring whenever $k = 1$ and $t_1 \geq 2$, or
iii) a SB-ring whenever $k = 1$ and $t_1 = 1$.

Remark. As an alternative approach to the validity of the above example, for any positive integer m which is not a power of a prime number and the fact that $Z_m[X]$ is a J-Noetherian ring since it is a Noetherian ring, we can apply Theorem 2.7 in [5], and Theorem 2.3 in [1], which is stated in the remark following Proposition 1.1 above together with $dim\, R + 1 \leq dim\, R[X] \leq 2dim\, R + 1$, to conclude that $Z_m[X]$ is a 2-stable ring which is not a B-ring.

The ring S in the following example, which is given by Dr. Marion E. Moore, provides an example of a 2-stable ring which is not a strongly 2_J-stable ring.

Example 3.2. Let R be the collection of all elements of the form $a\alpha + b\beta + c\gamma + d$ with $a, b, c, d \in Z_2$ where α, β, and γ satisfy the relations $\alpha^2 = \beta^2 = \gamma^2 = \alpha\beta = \beta\alpha = \alpha\gamma = \gamma\alpha = \beta\gamma = \gamma\beta = 0$ and $S = R \times R$. Note that since

S is a finite ring, then by Theorem 2.2 in [5] it is a B-ring and consequently a 2-stable ring. Now we show that S cannot be a strongly 2_J-stable ring. Clearly, $(0,\beta) \in ((1,\gamma),(0,\alpha),(0,\beta))$ and $(1,\gamma) \notin J(S)$ since $(1,1)-(1,\gamma)=(0,1-\gamma)$ is not a unit in S. Suppose that $(0,\beta) \in ((1,\gamma)+(r,s)(0,\beta),(0,\alpha)+(t,u)(0,\beta))$ for some $(r,s),(t,u) \in S$ where $r=r_0+r_1\alpha+r_2\beta+r_3\gamma$, $s=s_0+s_1\alpha+s_2\beta+s_3\gamma$, $t=t_0+t_1\alpha+t_2\beta+t_3\gamma$, and $u=u_0+u_1\alpha+u_2\beta+u_3\gamma$. Thus, $\beta=(\gamma+s\beta)f+(\alpha+u\beta)g$ for some f and g in R where $f=f_0+f_1\alpha+f_2\beta+f_3\gamma$ and $g=g_0+g_1\alpha+g_2\beta+g_3\gamma$. Consequently, $\beta=(\gamma+s\beta)f+(\alpha+u\beta)g=f_0\gamma+s_0f_0\beta+g_0\alpha+u_0g_0\beta=g_0\alpha+(s_0f_0+u_0g_0)\beta+f_0\gamma$ which implies $f_0=g_0=0$ and $s_0f_0+u_0g_0=1$. Therefore, $0=1$ which is a contradiction.

Example 3.3. Since R in the above example is a completely primary ring with nilpotent elements 0, α, β, γ, $\alpha+\beta$, $\alpha+\gamma$, $\beta+\gamma$, and $\alpha+\beta+\gamma$, then by Theorem 2.7 in [5], $R[X]$ is a B-ring and $S[X]$, $S=R\times R$ is not a B-ring. Now since every B-ring is a 2-stable ring and the homomorphic image of a strongly 2_J-stable ring is a strongly 2_J–stable ring, then by applying Theorem 2.4 and Example 3.2, it is clear that $S[X] \simeq R[X]\times R[X]$ is a 2-stable ring which is neither a B-ring nor a strongly 2_J-stable ring. Further, it is easy to show directly from the definition that every local ring, a ring with a unique maximal ideal, is a SB-ring. Consequently since every SB-ring is a strongly 2_J-stable ring, then $S=R\times R$ shows that the direct product of two strongly 2_J-stable rings need not be a strongly 2_J-stable ring. From this and the result in Theorem 2.4 that the product of strongly n-stable rings is again a strongly n-stable ring, we can conclude that the class of all strongly 2-stable rings is properly contained in the class of all strongly 2_J-stable rings. Note that also from the above argument, it is easy to see that R is a SB-ring which is not a strongly 2-stable ring.

Example 3.4. Every Boolean ring, a ring in which every element is an idempotent, is stable. Assume (a,b) is a unimodular ideal of a Boolean ring R. Thus, for some appropriate elements $x,y \in R$, $1=ax+by$. The result follows by multiplying both sides of this equation by $1-a$.

In the rest of this section we generalize some results of Section 8 in [1], namely, the necessary part of Proposition 8.2, the paragraph above Corollary 8.3, and the necessary part of Corollary 8.3.

Notation. Let t and s be two positive integers with $t \geq s \geq 2$ and let $d,a \in R$ with a not a unit in R, $\pi\colon R \to R/(a)$ the canonical epimorphism, $GL_d(R,s-1\times t)=\{\, M \in M(R,s-1\times t) \mid \ d$ is in the ideal generated by the determinants of all $s-1\times s-1$ submatrices of $M\,\}$, $SL_d(R,s\times s)=\{\, M \in M(R,s\times s) \mid$ the determinant of M is equal to $d\,\}$.

Theorem 3.1. If $SL_d(R, s \times s) \to SL_{\pi(d)}(R/(a), s \times s)$ is surjective, then $GL_d(R, s-1 \times s) \to GL_{\pi(d)}(R/(a), s-1 \times s)$ is surjective.

Proof. Let $M' \in GL_{\pi(d)}(R/(a), s-1 \times s)$, then there exists $\alpha' \in M(R/(a), 1 \times s)$ such that $\pi(d)$ is equal to the determinant of $\alpha' \times M'$ or equivalently $\alpha' \times M' \in SL_{\pi(d)}(R/(a), s \times s)$. Now, by hypothesis, we can lift $\alpha' \times M'$ to $\alpha \times M$. Since the determinant of $\alpha \times M$ is equal to d, then M is a member of $GL_d(R, s-1 \times s)$ and the proof is complete.

Note that Theorem 3.1 is a general form of the necessary part of Proposition 8.2 in [1]. We state this result below.

Corollary 3.1. If $SL(R, s \times s) \to SL(R/(a), s \times s)$ is surjective, then $GL(R, s-1 \times s) \to GL(R/(a), s-1 \times s)$ is surjective.

Proof. Apply Theorem 3.1 with $d = 1$.

We next generalize the result of the paragraph preceding Corollary 8.3 in [1].

Theorem 3.2. $GL_d(R, 1 \times s) \to GL_{\pi(d)}(R/(a), 1 \times s)$ is surjective if and only if for every ideal $(a_1, a_2, \ldots, a_s, a)$ of R containing d, there exist $b_1, b_2, \ldots, b_s \in R$ such that $d \in (a_1 + b_1 a, \ldots, a_s + b_s a)$.

Proof. For the necessary part let $d \in (a_1, a_2, \ldots, a_s, a)$, then $\pi(d) \in (a'_1, a'_2, \ldots, a'_s)$ where $a'_i = a_i + (a)$ for $1 \leq i \leq s$. Thus, $(a'_1, a'_2, \ldots, a'_s) \in GL_{\pi(d)}(R/(a), 1 \times s)$. Hence, by hypothesis, there exists $(\alpha_1, \alpha_2, \ldots, \alpha_s) \in GL_d(R, 1 \times s)$ such that $(\alpha_1, \alpha_2, \ldots, \alpha_s) \mapsto (\alpha'_1, \alpha'_2, \ldots, \alpha'_s)$. Thus, $\pi(\alpha_i) = \alpha_i + (a) = a_i + (a)$ which implies $\alpha_i = a_i + b_i a$ for $1 \leq i \leq s$ and $d \in (\alpha_1, \alpha_2, \ldots, \alpha_s) = (a_1 + b_1 a, a_2 + b_2 a, \ldots, a_s + b_s a)$. For the sufficiency if $(a'_1, a'_2, \ldots, a'_s)$ is a member of $GL_{\pi(d)}(R/(a), 1 \times s)$, then $\pi(d) \in (a'_1, a'_2, \ldots, a'_s)$. Hence, $\pi(d) = \sum_{i=1}^{s} \alpha'_i a'_i$ with $\alpha'_i \in R/(a)$. Thus, $d + (a) = \sum_{i=1}^{s} \alpha_i a_i + (a)$ which implies $d \in (a_1, a_2, \ldots, a_s, a)$. By hypothesis, $d \in (a_1 + b_1 a, a_2 + b_2 a, \ldots, a_s + b_s a)$ for some $b_1, b_2, \ldots, b_s \in R$. Hence, $(a_1 + b_1 a, a_2 + b_2 a, \ldots, a_s + b_s a)$ is a member of $GL_d(R, 1 \times s)$ and $(a_1 + b_1 a, a_2 + b_2 a, \ldots, a_s + b_s a) \mapsto (a'_1, a'_2, \ldots, a'_s)$.

Corollary 3.2. $GL(R, 1 \times s) \to GL(R/(a), 1 \times s)$ is surjective if and only if any unimodular sequence $(a_1, a_2, \ldots, a_s, a)$ is stable.

Proof. Apply Theorem 3.2 with $d = 1$.

Theorem 3.3. For $s \geq 2$, $GL(R, 1 \times s) \to GL(R/(a), 1 \times s)$ is surjective if and only if any unimodular sequence $(a_1, a_2, \ldots, a_s, a)$ in R with $(a_1, a_2, \ldots, a_{s-1}) \not\subseteq J(R)$ is stable.

Proof. The necessary part can be obtained from the necessary part of Corollary 3.2. For the sufficiency let $(a'_1, a'_2, \dots, a'_s)$ be a member of $GL(R/(a), 1 \times s)$. If $(a'_1, a'_2, \dots, a'_s) = R/(a)$, then there exists $1 \leq i \leq s$ such that $a'_i \notin J(R/(a))$. Without loss of generality, we can assume $i \neq s$. Thus, $(a'_1, a'_2, \dots, a'_{s-1}) \not\subseteq J(R/(a))$. So, $(a_1, a_2, \dots, a_{s-1}) \not\subseteq J(R)$ where $a'_i = a_i + (a)$ for $1 \leq i \leq s$. If $r_i \mapsto r'_i$, then $1 + (a) = \sum_{i=1}^{s} r'_i a'_i = \sum_{i=1}^{s} r_i a_i + (a)$ which implies $1 \in (a_1, a_2, \dots, a_s, a)$. Hence, by hypothesis, there exist $b_1, b_2, \dots, b_s \in R$ such that $1 \in (a_1 + b_1 a, a_2 + b_2 a, \dots, a_s + b_s a) \in GL(R, 1 \times s)$ and the proof is complete.

Theorem 3.4. Let $d \in R$. If $SL_d(R, 2 \times 2) \to SL_{\pi(d)}(R/(a), 2 \times 2)$ is surjective, then for any ideal (a_1, a_2, a) of R containing d there exist $b_1, b_2 \in R$ such that $d \in (a_1 + b_1 a, a_2 + b_2 a)$.

Proof. Apply Theorem 3.1 and Theorem 3.2.

Corollary 3.3. If $SL_a(R, 2 \times 2) \to SL_{\pi(a)}(R/(a), 2 \times 2)$ is surjective, then any sequence (a_1, a_2, a) with $a_1, a_2, a \in R$ is stable.

Proof. See Theorem 3.4.

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