# AREA OF A TRIANGLE IN TERMS OF THE TAXICAB DISTANCE

Münevver Özcan and Rüstem Kaya

**Abstract.** In this study, we use the taxicab distance function to calculate the area of a triangle, and give the taxicab version of Heron's Formula.

**1. Introduction.** The taxicab plane $\mathbb{R}_T^2$ is almost the same as the Euclidean analytical plane $\mathbb{R}^2$. The points are the same, the lines are the same, and angles are measured in the same way. However, the distance function is different. Taxicab distance between the points $P$ and $Q$ is the length of a shortest path from $P$ to $Q$ composed of line segments parallel to the coordinate axes. That is, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ then the taxicab distance from $P$ to $Q$ is $d_T(P, Q) = |x_1 - x_2| + |y_1 - y_2|$.

The taxicab plane geometry is non–Euclidean since it fails to satisfy the side–angle–side axiom but satisfies all the remaining twelve axioms of the Euclidean plane geometry [5]. Since the taxicab plane geometry has a different distance function it seems interesting to study the taxicab analogues of the topics that include the concept of distance in the Euclidean geometry. A few of such topics have been studied by some authors [1, 2, 3, 4, 6, 7, 9, 10]. The group of isometries that preserve taxicab distance is determined in [8].

Here, we study the following problem: How can one compute the area of a triangle in the analytical plane by using the taxicab distance? Clearly, a well–known formula

$$\text{area of a triangle} = (\text{base} \times \text{height})/2$$

is not, in general, valid in $\mathbb{R}_T^2$. It is valid if and only if the base is parallel to any one of the coordinate axes. (In this case, Euclidean and taxicab lengths of the base and height are the same.) The area of a triangle can also be computed using the three sides of the triangle. Let the sides of a triangle have lengths $a, b, c$. Introduce the semiperimeter $p = (a + b + c)/2$ and the area $A$. Then

$$A^2 = p(p - a)(p - b)(p - c),$$

is known as Heron's Formula. The aim of this work is to give a taxicab version of Heron's Formula.

**2. Extension of Heron's Formula to the Taxicab Plane.** Let the sides of a triangle $ABC$, in the taxicab plane, have lengths $a_T = d_T(B,C)$, $b_T = d_T(A,C)$ and $c_T = d_T(A,B)$, and denote the *taxicab–semiperimeter* $p_T = (a_T + b_T + c_T)/2$. The following two propositions give the taxicab versions of Heron's formula in some special cases.

<u>Proposition 1</u>. If one side of a triangle $ABC$, say $BC$, is parallel to one of the coordinate axes and none of the angles $B$ and $C$ is an obtuse angle, then for the area $A$ of $ABC$,

$$A = \frac{1}{2}a_T(p_T - a_T) = \frac{1}{4}a_T(b_T + c_T - a_T).$$

<u>Proof</u>. Consider the triangle $ABC$. Let $h_T = d_T(A, BC)$ and $a'_T = d_T(B, A')$, where $A'$ denotes the foot of the altitude from A, (Figure 1).

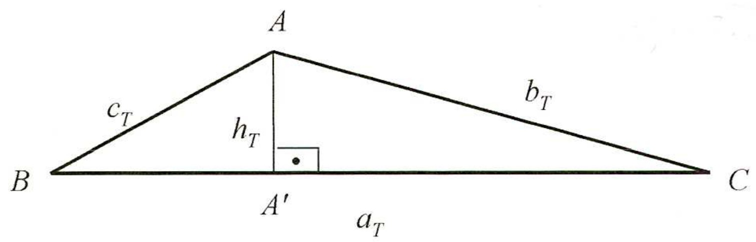


Figure 1. $BC$ is parallel to a coordinate axis. $B$ and $C$ are not obtuse angles.

For the triangles $ABA'$ and $AA'C$ we get

$$a'_T = c_T - h_T \quad \text{and} \quad b_T = h_T + (a_T - a'_T),$$

respectively. From these equalities;

$$b_T = h_T + a_T - c_T + h_T \quad \text{and} \quad h_T = (b_T + c_T - a_T)/2,$$

and consequently,

$$A = \frac{1}{2}a_T h_T = \left(\frac{a_T}{2}\right)\left(\frac{-a_T + b_T + c_T}{2}\right).$$

Using the taxicab–semiperimeter $p_T$, in the above statement one obtains,

$$A = \frac{1}{2} a_T (p_T - a_T)$$

which completes the proof. Unfortunately, in all the remaining cases, one needs a new distance parameter $a'_T$ to determine the area of a given triangle in terms of the taxicab lengths of sides.

Proposition 2. If one side of a triangle $ABC$, say $BC$, is parallel to one of the coordinate axes and one of the angles $B$ and $C$ is not an acute angle, then for the area $A$ of $ABC$

$$A = \frac{1}{2} a_T (p_T - (a_T + a'_T)),$$

where $a'_T = d_T(A', B)$ or $a'_T = d_T(A', C)$ depending on whether $B$ or $C$ is not an acute angle, respectively; and $A'$ denotes the foot point of the altitude from $A$.

Proof. Consider any of the triangles $ABC$ in Figure 2a or Figure 2b and let $d_T(A, A') = h_T$, where the side $BC$ is parallel to one of the coordinate axes.

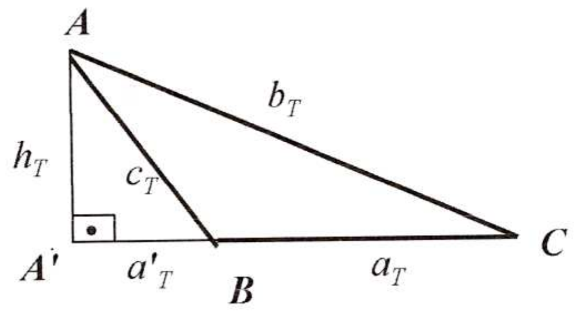


Figure 2a. $BC$ is parallel to a coordinate axis and $B$ is not an acute angle.

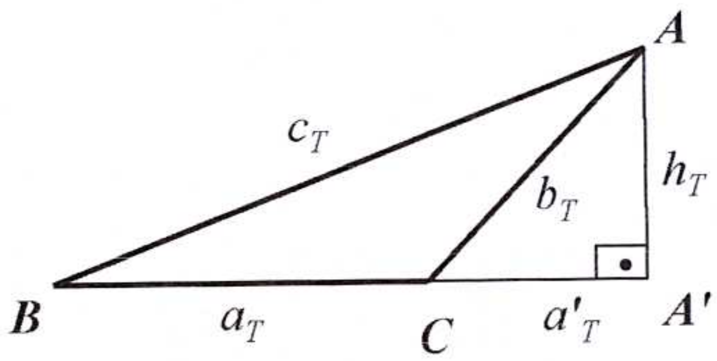


Figure 2b. $BC$ is parallel to a coordinate axis and $C$ is not an acute angle.

It is easily seen, from the triangles $AA'C$ and $AA'B$ that

$$h_T = b_T - (a_T + a'_T) \text{ and } h_T = c_T - a'_T$$

if $B$ is not an acute angle. Thus, $b_T = a_T + c_T$. Then for the area $A$ of the triangle $ABC$,

$$A = \frac{1}{2}a_T h_T = \frac{1}{2}a_T(b_T - (a_T + a'_T)) = \frac{1}{2}a_T\left(\frac{b_T + b_T}{2} - (a_T + a'_T)\right)$$

$$= \frac{1}{2}a_T\left(\frac{a_T + b_T + c_T}{2} - (a_T + a'_T)\right).$$

Using $p_T$ we obtain, $A = \frac{1}{2}a_T(p_T - (a_T + a'_T))$. Similarly, in the case where $C$ is not an acute angle then,

$$h_T = c_T - (a_T + a'_T) \quad \text{and} \quad h_T = b_T - a'_T$$

from the triangles $ABA'$ and $ACA'$. Thus, $c_T = b_T + a_T$. Now, it can be easily shown that the above formula is still valid.

Corollary. Let the side $BC$ of a triangle $ABC$ in the taxicab plane, be parallel to one of the coordinates axes. Then if the angle $B \geq \pi/2$ or $C \geq \pi/2$ then

$$b_T = a_T + c_T \quad \text{or} \quad c_T = a_T + b_T,$$

respectively.

Now, we introduce some new concepts in order to find a general taxicab version of Heron's Formula for triangles such that none of their sides is parallel to the coordinate axes. Let $ABC$ be any triangle in the taxicab plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to a coordinate axis. $A$ line $l$ is called a *base line* of $ABC$ if and only if

1. $l$ passes through a vertex,
2. $l$ is parallel to a coordinate axis,
3. $l$ intersects the opposite side (as a line segment) to the vertex in Condition 1.

Clearly, at least one of the vertices of the triangle always has one or two base lines. Such a vertex of a triangle is called a *basic vertex*. A *base segment* is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

The following theorem gives the general taxicab version of Heron's Formula.

Theorem 3. Let $\alpha_T$ denote the length of a base segment of a triangle. Then for the area $A$ of a triangle described above,

$$A = \begin{cases} \alpha_T(p_T - (\alpha_T + \alpha'_T))/2 & \text{, if there exists only one base line} \\ & \quad \text{passing through the basic vertex} \\ \alpha_T(p_T - (\alpha_T + \alpha'_T + \alpha''_T))/2 & \text{, if there exist two base} \\ & \quad \text{lines passing through the basic vertex} \end{cases}$$

where $a'_T = d_T(D, H), \alpha''_T = d_T$ (basic vertex, $H'$) and

D = Intersection point of the base line and the opposite side,

H = The point of orthogonal projection of one of the remaining two vertices on the base line but not on the base segment,

H′= The point of orthogonal projection of the third vertex on the same base line but not on the base segment.

Proof. Let $ABC$ be a triangle with the basic vertex $C$, and $l_C$ denote a base line through $C$. Then $D = l_C \cap [AB]$ where $[AB]$ is the side $AB$ of triangle $ABC$. Let $H$ be the orthogonal projection of the vertex $B$ to $l_C$ but not on $[CD]$. This choice of $B$ is always possible since $A$ and $B$ have symmetrical roles (see Figures 3 and 4). Now, the triangle $ABC$ is a combination of the triangles $ADC$ and $BDC$. If $c'_T = d_T(A,D)$ and $c''_T = d_T(B,D)$ then the taxicab–semiperimeters of $ADC$ and $BDC$ are

$$p'_T = (\alpha_T + b_T + c'_T)/2 \quad \text{and} \quad p''_T = (\alpha_T + a_T + c''_T)/2,$$

respectively. Consequently, using $p_T = (a_T + b_T + c_T)/2$,

$$p'_T + p''_T = p_T + \alpha_T.$$

Now two cases are possible.

i. If $l_C$ is the only base line through the basic vertex $C$ (Figure 3) then,

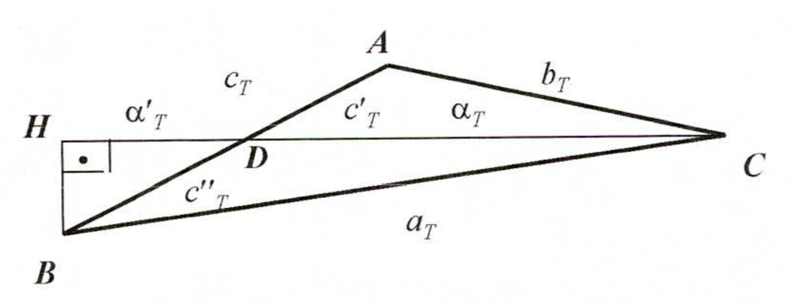


Figure 3. A triangle $ABC$ with a base line on the base vertex

the area $A_1$ of $ADC$ is given by

$$A_1 = \alpha_T(p'_T - \alpha_T)/2$$

by Proposition 1, and the area $A_2$ of $BDC$ is given by

$$A_2 = \alpha_T(p''_T - (\alpha_T + \alpha'_T))/2$$

by Proposition 2. Then using the equality $p'_T + p''_T = p_T + \alpha_T$,

$$\begin{aligned} A = A_1 + A_2 &= (\alpha_T(p'_T - \alpha_T)/2) + (\alpha_T(p''_T - (\alpha_T + \alpha'_T))/2) \\ &= \alpha_T((p'_T + p''_T) - \alpha_T - (\alpha_T + \alpha'_T))/2 \\ &= \alpha_T(p_T - (\alpha_T + \alpha'_T))/2. \end{aligned}$$

ii. If there exist two base lines through the basic vertex $C$ (Figure 4) then,

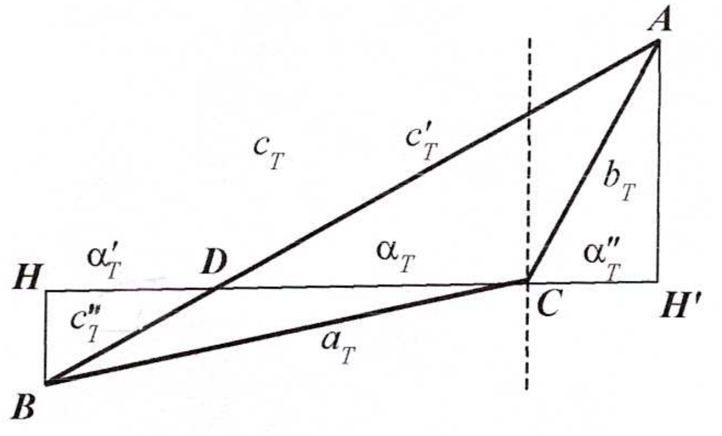


Figure 4. A triangle $ABC$ with two base lines on the base vertex

the area $A_1$ of $ADC$ is given by

$$A_1 = \alpha_T(p'_T - (\alpha_T + \alpha''_T))/2$$

and the area $A_2$ of $BDC$ is given by

$$A_2 = \alpha_T(p''_T - (\alpha_T + \alpha'_T))/2$$

by Proposition 2. Then using the equality $p'_T + p''_T = p_T + \alpha_T$,

$$\begin{aligned}
A = A_1 + A_2 &= (\alpha_T(p'_T - (\alpha_T + \alpha''_T))/2) + (\alpha_T(p''_T - (\alpha_T + \alpha'_T))/2) \\
&= \alpha_T((p'_T + p''_T) - (\alpha_T + \alpha''_T) - (\alpha_T + \alpha'_T))/2 \\
&= \alpha_T(p_T - (\alpha_T + \alpha'_T + \alpha''_T))/2.
\end{aligned}$$

Notice that here $\alpha_T + \alpha'_T + \alpha''_T = d_T(H, H')$.

*References*


1. Z. Akça, and R. Kaya, "On the Taxicab Trigonometry," *Jour. of Inst. of Math. and Comp. Sci. (Math. Ser)*, 10 (1997), 151–159.
2. C. Ekici, I. Kocayusufoğlu, and Z. Akça, "The Norm in Taxicab Geometry," *Tr. J. of Mathematics*, 22 (1998), 295–307.
3. Y. P. Ho and Y. Liu, "Parabolas in Taxicab Geometry," *Missouri J. of Math. Sci.* 8 (1996), 63–72.
4. R. Kaya, Z. Akça, .I. Günaltili, and M. Özcan, "General Equation for Taxicab Conics and Their Classification," *Mitt. Math. Ges Hamburg*, 19 (2000), 1–14.
5. E. F. Krause, *Taxicab Geometry*, Addison – Wesley, Menlo Park, 1975.
6. R. Laatsch, "Pyramidal Sections in Taxicab Geometry," *Mathematics Magazine*, 55 (1982), 205–212.
7. M. Özcan and R. Kaya, "On the Ratio of Directed Lengths in the Taxicab Plane and Related Properties," *Missouri J. of Math. Sci.*, 14 (2002), 107–117.
8. B. E. Reynolds, "Taxicab Geometry," *Pi Mu Epsilon Journal*, 7 (1980), 77–88.
9. D. J. Schattschneider, "The Taxicab Group," *Amer. Math. Monthly*, 91 (1984), 423–428.
10. S. S. So and Z. S. Al–Maskari, "Two Simple Examples in Non-Euclidean Geometry," *Kansas Science Teacher (J. of Math. and Science Teaching)*, 11 (1995), 14–18.
11. S. Tian, S. S. So, and G. Chen, "Concerning Circles in Taxicab Geometry," *Int. J. Math. Educ. Sci. Tech.*, 28 (1997), 727–733.



Münevver Özcan
Department of Mathematics
University of Osmangazi
26480, Eskişehir
Turkey
email: mozcan@ogu.edu.tr

Rüstem Kaya
Department of Mathematics
University of Osmangazi
26480, Eskişehir
Turkey
email: rkaya@ogu.edu.tr