BETWEEN CONSECUTIVE SQUARES

Richard L. Francis

The mystery of the distribution of the primes continues in its challenge to mathematicians as the twenty-first century unfolds. Among its many unfinished chapters is the tantalizing question surrounding the seeming occurrence of primes between any two squares.

The pioneering work of P. L. Tchebychef (1821–1894) in the mid-nineteenth century, building on the notes of Adrien Marie Legendre (1752–1833) and Carl Friedrich Gauss (1777–1855), was to set the stage for an ultimate and rigorous disposition of the Prime Number Theorem. Not only would he resolve Bertrand's Conjecture in the affirmative and thus provide another look at the infinitude of the primes, so too would he provide a deeper look at $\pi(x)$. Such a symbol denotes the number of primes less than or equal to x. This early analysis of $\pi(x)$ bordered closely on establishing the key limit result itself, namely,

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}} = 1.$$

Such a theorem would find its resolution in the simultaneous discoveries of Jacques Hadamard (1865–1963) and Charles de la Vallée Poussin (1866–1962). The year of discovery was 1896. Though the theorem speaks of the number of primes only in an approximate manner, it provides a powerful basis for various conjectures as diverse number classes are considered.

Speculation. Paralleling the definition of $\pi(x)$, let $\alpha_n(x)$ represent the number of nth powers less than or equal to x. As $\pi(x) > \alpha_2(x)$ for all x sufficiently large, it is tempting to conclude that between any two squares, there exists a prime. Note that $\pi(x) \approx \frac{x}{\ln x}$ for large values of x and that $\alpha_2(x) = [\sqrt{x}] \approx \sqrt{x}$. As $\ln x < \sqrt{x}$, then $\frac{x}{\ln x} > \sqrt{x}$. Capitalizing on the fact that

$$\pi(x) > \frac{x}{\ln x} > \sqrt{x} \geq [\sqrt{x}],$$

it follows that the number of primes exceeds that of the squares as the reference number x becomes large.

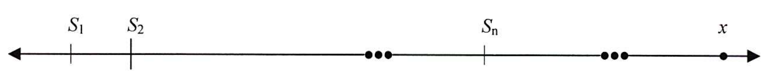
It appears doubtful that this super-abundance of primes can be clustered in such a way so as to avoid appearing at least once between consecutive squares. Some suggestion is provided by the table below.

A Prime-Square Table

X	$\pi(x)$	$\alpha_2(x)$
10	4	3
20	8	4
100	25	10
200	46	14
1000	168	31
2000	303	44
10000	1229	100
100000	9592	316
1000000	78498	1000
1000000000	50847534	31622
1000000000000	37607912018	1000000
10000000000000000	279238341033925	100000000
...	...	...

A further suggestion of the occurrence of primes between any two consecutive squares relates to the number line.

Squares on the Number Line



Let $S_1, S_2, \ldots, S_n$ denote all the squares less than or equal to x. As there are many more primes, namely, $\pi(x)$, in [1, x] than consecutive square intervals or slots, a random scattering of these primes strongly suggests at least one prime per slot. Consider, say, an x value of one million. The probability appears very favorable in distributing 78498 primes in only a thousand slots that at least one prime should appear in each.

Though such an outcome seems reasonable in a probabilistic sort of way, no rigorous demonstration has today been given of this Prime-Square Betweenness Conjecture (PSBC). Obviously, any prime must appear between consecutive squares, namely the squares immediately before and after it, a fact which implies the truth of the outcome in infinitely many cases. But, are consecutive square intervals bypassed in the distributing of the primes?

Number Groupings. Number groupings prove of interest in speculating about prime distributions. For example, centuries denote groupings by hundreds and begin with the first 100 positive integers. The *first* primeless century begins with 1671801 and extends through 1671900. Yet no consecutive squares appear in this interval, only the lone square 1671849. Accordingly, no counterexample to the prime-square betweenness conjecture here occurs.

Similarly, the first encounter with two consecutive primeless centuries is the grouping beginning with 191912801 and extending through 191913000. No squares at all appear in this two-century set.

Can the process now be extended to three consecutive primeless centuries? And four? Or to millennia and even higher groupings? Could such a vast *primeless* grouping eventually be found so as to include two consecutive squares?

An interesting and similar construction relates to composite chains of any length whatsoever. This well-known technique of generating such primeless chains involves the interval from $n! + 1$ to $n! + n$. All such numbers are of necessity composite with the possible exception of $n! + 1$. This number proves composite only as $n + 1$ is prime (i.e., Wilson's Theorem). The problem reduces basically to the occurrence of consecutive squares in such an interval. Should, for example, the interval from $100! + 1$ to $100! + 100$ contain two consecutive squares, then an identification would immediately follow of consecutive squares with no primes between them.

Squares Between $\mathbf{n!+a}$ And $\mathbf{n!+b}$. Consider thus the conjecture that $n!+a$ and $n!+b$ cannot be consecutive squares if a and $b (a < b)$ are between 1 and n. If so, and letting $n! + a = x^2$, $n! + b = (x + 1)^2$, then $b - a = 2x + 1$. A fairly straightforward process shows the truth of the conjecture.

(1) $b - a = 2x + 1 = 2\sqrt{n! + a} + 1$.
(2) $n \geq b$.
(3) As $b = 2\sqrt{n! + a} + a + 1$, then $n \geq 2\sqrt{n! + a} + a + 1$.
(4) Yet, $n^2 < n!$ for $n > 3$.
(5) So $n \leq \sqrt{n!}$ thus contradicting step 3 above.

Accordingly, this familiar method of constructing finite primeless chains of any length whatever does not generate a consecutive square interval of no primes. Lone squares may appear anywhere from the beginning ($7!+1 = 71^2$) to the end ($3!+3 = 3^2$). Moreover, squares may not occur at all ($6! + 1$, etc.).

It is easy to establish that there are infinitely many millennia devoid of primes by the factorial procedure. Note that the millennium from $1000000! + 1001$ to $1000000! + 2000$ consists of no primes. More impressively the lengthy chain of integers from $1000000! + 2$ to $1000000! + 1000000$ contains 999,999 consecutive integers none of which are prime. In spite of lengthy intervals of incredibly large but consecutive composites, no counterexamples can there be found in rejecting the Prime-Square Betweenness Conjecture.

It might be conjectured that the farther out the century (or millennium, etc.), the smaller each time the number of primes. This would clearly violate the infinitude of the primes because of the primeless groupings noted here. For example, the first grouping of ten trillion positive integers contains 346,065,535,898 primes. Yet a certain later grouping of ten trillion positive integers will contain no primes. Still later groupings will again contain primes.

A Note On Higher Powers. By an analysis similar to the above, it can be shown that no two consecutive *cubes* can appear in the interval from $n!+1$ to $n!+n$. The proof hinges on the fact that $n^3 \leq n!$ for $n > 5$, accompanied by a case by case verification of the absence of consecutive cubes where $n \leq 5$. Extension is quickly made to fourth powers and from there on in general by paralleling the square and cubic arguments. Obviously, consecutive fourth powers are not consecutive squares.

Significantly, any interval with consecutive cubes as endpoints necessarily contains an interval with consecutive squares as endpoints. Should it be established that a cubic interval containing no primes exists, then it must follow that a square interval containing no primes also exists. In essence, if the Prime-Cube Betweenness Conjecture can be disproved, a rejection of the Prime-Square Betweenness Conjecture would immediately follow. Generally, rejecting the Prime-nth Power Betweenness Conjecture also rejects the Prime-kth Power Betweenness Conjecture for all $k \leq n$. Abbreviated in symbols, $\sim n \rightarrow \sim k$, or, by the contrapositive, $k \rightarrow n$.

Some Unsolved Problem Connections. Either element of a set of prime twins obviously must appear between some two consecutive squares. This simple fact moreover implies that both of the primes belong to the same interval. Obviously, p and $p+2$ can be in different square intervals only as a square lies between them. Hence, $p = x^2 - 1$ (or $p+2 = x^2+1$). As $x^2 - 1$ is algebraically factorable, it follows that a given set of prime twins belongs to the same consecutive square interval. An unsolved problem relating to this is that of the conjecture "the set of prime twins is an infinite set." Any quest for prime twins can thus be narrowed down to intervals whose endpoints are consecutive squares. It is possible for more than one set of prime twins to appear between consecutive squares. For example, the pair 101 and 103 and the pair 107 and 109 appear between 10^2 and 11^2. An extended form of the PSBC is that between consecutive squares, two primes will always appear. Perhaps this can be extended to more than two beyond a certain point in the progression of integers.

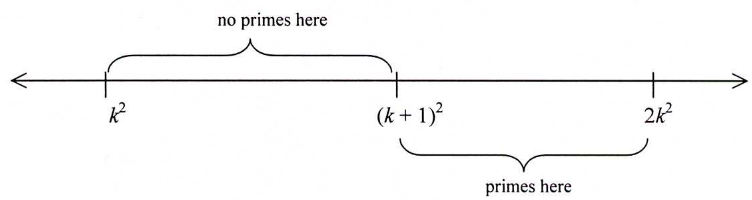
Relatedly, no interval with consecutive square endpoints can contain more than one of certain prime number types. These types include Fermat primes, Mersenne primes, and repunit primes. None of these sets are known to be infinite. The spacing between consecutive primes in such categories is so vast that ever enlarging consecutive square intervals prove insufficient in length to allow for two or more such primes.

Suppose, for example, that the Fermat primes $2^x + 1$ and $2^y + 1$ are between consecutive squares (where $x < y$). As x and y are necessarily powers of 2, then 2^x is the square to the immediate left of $2^x + 1$. Moreover, 2^y is the square immediately to the left of $2^y + 1$. Hence, these two consecutive Fermat primes cannot both occur between the square 2^x and the square which immediately follows it. A similar argument follows for Mersenne primes, that is, those of the form $2^x - 1$. Here, x would necessarily be prime.

Should R_p denote a repunit prime (thus consisting of p ones in its representation), p would of necessity be prime. The closest repunit larger than R_p would be R_q where $q = p + 2k, k \geq 1$. So between R_p and R_q, all integers of $p + 1$ digits appear. Yet there are at least two squares of any select number of digits. Essentially, these two consecutive repunit primes cannot both appear between consecutive squares.

The reader is invited to consider a similar analysis for other prime types. These include isolated primes, Euclidean primes, primes of the form $n! + 1$, and more. Set cardinality questions arise in all of these.

A Mathematical Aside. In spite of the inductive evidence to the contrary, suppose the Prime-Square Betweenness Conjecture fails, not once, but infinitely many times. Thus, it becomes easy to show that between $(n + 1)^2$ and $2n^2$, there is a prime for infinitely many values of n.



By Bertrand's (Conjecture Theorem), there is at least one prime between any integer greater than 1 and its double. It is easy to verify by a quadratic inequality that $(n+1)^2 < 2n^2$ for n greater than or equal to 3. Consider any interval from k^2 to $(k+1)^2$ for which the Prime-Square Betweenness Conjecture fails. Then the interval from $(k+1)^2$ to $2k^2$ must contain a prime (as guaranteed by Bertrand's Theorem). There are infinitely many such intervals according to the hypothesis. This problem gives rise to an interesting extension (higher powers of n) as well as the challenging question concerning what's between consecutive powers of the integers.

Moreover, Bertrand's Conjecture, if generalized, reveals a valid theorem which contains a lesser value of the multiplier. That is,

> "For any $\epsilon > 0$ and n sufficiently large, there exists a prime between n and $(1+\epsilon)n$."

Note the reduction of the multiplier of n with its original coefficient 2 now replaced by $1+\epsilon$. Hence, sufficiently far out in the sequence of integers, mathematicians may confidently assert the existence of a prime between any such number n and $(1 + 10^{-100})n$. Such a late modern era revelation hinges on the Prime Number Theorem, its notable function $\pi(x)$, and the previously mentioned works of Hadamard and Poussin.

References

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Richard L. Francis
Department of Mathematics
Southeast Missouri State University
Cape Girardeau, MO 63701