FUNCTIONS AS SUMS OF ODD AND EVEN FUNCTIONS

G. Alan Cannon and Kirby Smith

1. Introduction. It is well-known that every function mapping the real numbers into the real numbers can be written as the sum of an odd function and an even function. One only needs to use the additive properties of the real numbers to show this. What other groups have the property that every function on the group can be written as the sum of an odd function and an even function? To answer this, we first need to establish our basic definitions.

Let $(G,+)$ be a group, written additively, but not necessarily abelian. A function $f: G \to G$ is *odd* if $f(-x) = -f(x)$ for all $x \in G$ and *even* if $f(-x) = f(x)$ for all $x \in G$. We say the function $f: G \to G$ *splits* if f can be written as the sum of an odd function and an even function on G. Recall that a group has finite exponent if the least common multiple of the orders of all the elements of G is finite. Certainly every finite group has finite exponent, but the converse is not true. For example, the group of polynomials $(\mathbb{Z}_m[x],+)$ is infinite with finite exponent. In this paper we determine all groups G with finite exponent for which every function on G splits.

2. Main Results. Certainly if $G = \{0\}$, then every function on G splits since the zero function is both even and odd. We also can quickly identify a class of groups over which every function will split.

Lemma 1. Let G be a 2-group of exponent 2. Then every function on G can be written as the sum of an odd function and an even function.

Proof. If G has exponent two, then $a + a = 0$ and $a = -a$ for all $a \in G$. So if $f: G \to G$ is a function and $a \in G$, then $f(a) = f(-a)$ and f is an even function. Since the zero function 0 is odd, then $f = 0 + f$ and f splits.

In the case that G is a group of exponent two, the representation of a function $f: G \to G$ as the sum of an odd function and an even function is not unique. In the proof above, we showed that $f = 0 + f$ where 0 is an odd function and f is an even function. But since $x = -x$ for all $x \in G$, then $f(-a) = f(a) = -f(a)$ and f is an odd function. Since the zero function 0 is even, then $f = f + 0$ and f splits in two different ways.

Henceforth, we assume that G is a group with at least one element of order greater than two.

We say $g \in G$ is (*uniquely*) *halvable* if $g = x + x$ for some (unique) $x \in G$, and the group G is (*uniquely*) *halvable* if every element in G is (uniquely) halvable.

We now state our main result.

Theorem 2. Let G be a group with finite exponent and at least one element of order greater than two. The following are equivalent:

(i) Every function $f\colon G \to G$ can be written uniquely as the sum of an odd function and an even function;

(ii) G is halvable;

(iii) G has no element of order two;

(iv) G is uniquely halvable.

Proof. Assume property (i) holds. Since we are assuming that G has an element of order greater than two, we can fix $m \in G$ such that $m \neq -m$. Let $g \in G$ and define $f_g\colon G \to G$ by

$$f_g(x) = \begin{cases} g, & \text{if } x = m \\ 0, & \text{if } x \neq m. \end{cases}$$

The function f_g splits by assumption. Hence, $f_g = O + E$ for some odd function O and some even function E. Then $f_g(-x) = O(-x) + E(-x) = -O(x) + E(x)$ for all $x \in G$. Taking inverses of both sides yields $-f_g(-x) = -E(x) + O(x)$. Adding this last expression to $f_g(x) = O(x) + E(x)$ gives $f_g(x) - f_g(-x) = O(x) + O(x)$. Thus, $f_g(m) - f_g(-m) = O(m) + O(m)$ and $g = O(m) + O(m)$. So g is halvable. Since $g \in G$ is arbitrary, G is halvable and (ii) has been proved.

Now let G be a halvable group and suppose that G has an element of order two, hence elements of even order. Since G has finite exponent, there is an element $a \in G$ of maximum even order. Since a is halvable, then there exists $b \in G$ such that $b + b = a$. Thus, $|b| = 2|a|$, $|b|$ is even, and $|b| > |a|$, a contradiction. So G has no element of order two, and we have property (iii).

Assume property (iii) is true. We follow the proof of Theorem 2 of [1]. Let $g \in G$ with $|g| = t$. Since G has no element of order 2, then t is odd and $\frac{t+1}{2}$ is an integer. Consider $\left(\frac{t+1}{2}\right)g \in G$. Then

$$\left(\frac{t+1}{2}\right)g + \left(\frac{t+1}{2}\right)g = (t+1)g = tg + g = 0 + g = g$$

and g is halvable. So G is halvable.

To show uniqueness, assume g has another half $h \in G$. Then $h + h = g$. Let $|h| = s$. Then $sh = 0$ and $h + (s-1)h = 0$. So

$$h = (s-1)(-h) = \left(\frac{s-1}{2}\right)(-2h) = \left(\frac{s-1}{2}\right)(-g).$$

Thus, $h \in \langle g \rangle$, the cyclic subgroup generated by g. But g has only one half in $\langle g \rangle$, namely $\left(\frac{t+1}{2}\right)g$. So $h = \left(\frac{t+1}{2}\right)g$ and g is uniquely halvable. Property (iv) now follows.

Assume G is uniquely halvable. Then we can define $\left(\frac{1}{2}\right)a = x$ if and only if $x + x = a$. Let $f\colon G \to G$ be a function. Define $O\colon G \to G$ by

$$O(x) = \left(\frac{1}{2}\right)\big[f(x) - f(-x)\big]$$

for all $x \in G$. To show that O is odd, note that $O(x) + O(x) = f(x) - f(-x)$ for all $x \in G$. Taking inverses of both sides gives $-O(x) - O(x) = f(-x) - f(x)$. But $O(x) + O(x) = f(x) - f(-x)$ also implies that $O(-x) + O(-x) = f(-x) - f(x)$. Hence, $-O(x) - O(x) = O(-x) + O(-x)$. By uniqueness of halvability, $-O(x) = O(-x)$ and O is odd.

Define $E\colon G \to G$ by $E(x) = -O(x) + f(x)$ for all $x \in G$. Then $E(-x) = -O(-x) + f(-x) = O(x) + f(-x)$. From above we know that $O(x) + O(x) = f(x) - f(-x)$. Rearranging terms yields $O(x) + f(-x) = -O(x) + f(x)$. Thus, $E(-x) = E(x)$ and E is an even function. It is clear that $O(x) + E(x) = f(x)$, so f splits.

Suppose that a function on G splits in two different ways, say $O_1 + E_1 = O_2 + E_2$ for some odd functions O_1 and O_2 and some even functions E_1 and E_2. Then $-O_2 + O_1 = E_2 - E_1$. It follows that for all $x \in G$, $O_2(x) - O_1(x) = -O_2(-x) + O_1(-x) = (-O_2 + O_1)(-x) = (E_2 - E_1)(-x) = E_2(-x) - E_1(-x) = E_2(x) - E_1(x) = (E_2 - E_1)(x) = (-O_2 + O_1)(x) = -O_2(x) + O_1(x)$. Rearranging terms in the first and last expressions in the string of equalities above gives $O_2(x) + O_2(x) = O_1(x) + O_1(x)$. Since G is uniquely halvable, then $O_2(x) = O_1(x)$ for all $x \in G$. So $E_1(x) = E_2(x)$ for all $x \in G$. Therefore, every function on G splits uniquely. Thus, (iv) implies (i) and the proof is complete.

Notice that if G is the real numbers, then the functions O and E in the proof above are the two functions used to split any function over the reals. If G is finite and every function on G uniquely splits, then condition (iii) of the theorem implies that G has odd order. Furthermore, the proof of Theorem 2 actually only requires that G be a torsion group and that there is a bound on the elements of G of even order.

In the uniqueness part of the proof of (iv) implies (i) above, we showed that the sum of two even functions is again even. However, since G is not necessarily abelian, then the sum of two odd functions is not necessarily odd.

Using the previous theorem, we can answer the question posed at the beginning of the paper.

Corollary 3. Let G be a nontrivial group with finite exponent. Then every function from G to G can be written as the sum of an odd function and an even function if and only if G is a 2-group of exponent 2 or G has no element of order two.

Reference

1. B. C. McQuarrie and J. J. Malone, "Endomorphism Rings of Non-Abelian Groups," *Bull. Austral. Math. Soc.,* 3 (1970), 349–352.

G. Alan Cannon
Department of Mathematics
Southeastern Louisiana University
Hammond, LA 70402
email: acannon@selu.edu

Kirby Smith
Department of Mathematics
Texas A & M University
College Station, TX 77843
email: ksmith@math.tamu.edu