A COMMON FIXED POINT THEOREM VIA AN APPROACH BY JOSEPH AND KWACK

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Recently in [1], J. E. Joseph and M. H. Kwack introduced an alternative approach to proofs of fixed point theorems of contraction type. The approach is different from the usual proofs, which are variations of Banach's ingenious proof of his celebrated Contraction Mapping Theorem. Joseph and Kwack made the case that such a method of attack is one that a student in an analysis course would likely use if left to his or her own devices. The purpose of this note is to show that their approach produces a nice proof of the following theorem on maps with common fixed points. The flavor of the usual approach can be found in papers referenced in [1].

Theorem. Let (X, d) be a complete metric space, let g, h be selfmaps of X, and let $0 \leq \mu < 1/2$ such that

$$d(g(x),h(y)) \leq \\ \mu \max\{d(x, g(x)), d(y, g(x)), d(x, h(y)), d(y, h(y)), d(x, y)\}$$

for all $x, y \in X$. Then g and h have a common unique fixed point.

Proof. It is clear that if g and h have a common fixed point, that point is unique. Utilizing the notation from [1], if f is a selfmap on X, we let $I(f) = \{d(x, f(x)) \ : \ x \in X\}$, and let $c = \inf(I(g) \cup I(h))$. Because of symmetry we need show only that g and h have a common fixed point when $c = \inf I(g)$. We show first that $c = 0$. If not, since $p = \mu/(1 - \mu) < 1$, we have $c/p > c$; choose an $x \in X$ such that $pd(x, g(x)) < c$. Then x satisfies

$$d(g(x),h(g(x))) \leq \\ \mu \max\{d(x, g(x)), d(g(x), h(g(x))), d(x, h(g(x))), d(x, g(x))\}.$$

So $d(g(x), h(g(x))) \leq pd(x, g(x)) < c$, a contradiction. Let x_n be a sequence in X such that $d(x_n, g(x_n)) \to 0$. It follows from

$$d(x_n, h(x_n)) \leq \frac{1 + \mu}{1 - \mu} d(x_n, g(x_n))$$

that $d(x_n, h(x_n)) \to 0$. Now

$$d(x_n, x_m) \le d(x_n, g(x_n)) + d(g(x_n), h(x_m)) + d(h(x_m), x_m)$$

and

$$d(g(x_n), h(x_m)) \le \mu(d(x_n, g(x_n)) + d(x_m, h(x_m))) + d(x_n, x_m)).$$

Hence,

$$d(x_n, x_m) \le \frac{1+\mu}{1-\mu}(d(x_n, g(x_n)) + d(h(x_m), x_m)).$$

So x_n is Cauchy. Choose $v \in X$ such that $x_n \to v$. Then $g(x_n) \to v$ and $h(x_n) \to v$. From the inequalities

$$\begin{aligned} d(g(x_n), h(v)) \le & \\ & \mu \max\{d(x_n, g(x_n)), d(v, g(x_n)), d(x_n, h(v)), d(v, h(x_n)), d(x_n, v)\} \\ d(h(x_n), g(v)) \le & \\ & \mu \max\{d(x_n, h(x_n)), d(v, h(x_n)), d(x_n, g(v)), d(v, g(x_n)), d(x_n, v)\} \end{aligned}$$

it follows that $g(v) = h(v) = v$.

Reference

1. J. E. Joseph and M. H. Kwack, "Alternative Approaches to Proofs of Contraction Mapping Fixed Point Theorems," *Missouri Journal of Mathematical Sciences,* 11 (1999), 167–175.

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