UTILIZING THE EXPANSION OF $P^n - Q^n$ TO INTRODUCE AND DEVELOP THE EXPONENTIAL FUNCTION

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Recently, Bayne et al. [1, 2], have applied the identity

$$P^n - Q^n = (P - Q)\sum_{k=0}^{n-1} P^k Q^{n-1-k} \tag{1}$$

for real P, Q and positive integers n to present simple proofs of the existence of nth roots and inequalities used in real analysis. In this article the identity (1) is used to prove that f defined by

$$f(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

is a real–valued continuous function onto the positive reals with the collection of reals as its domain, and to establish some properties of f, including $f(x+y) = f(x)f(y)$, $f(0) = 1$ and an elegant proof that $f' = f$ where f' represents the derivative function for f. The equation $f(r) = (f(1))^r$ is shown to hold for rational r. This motivates the notation $f(x) = (f(1))^x = e^x$ and calling f the exponential function.

As in [4], the exponential function is often introduced as the inverse of the logarithmic function which is defined as

$$\int_1^x \frac{1}{t}dt.$$

Later, when convergence of sequences is studied, e^x is proved to be the limit of the sequence $(1 + \frac{x}{n})^n$. There again the logarithmic function is used. Dieudonné [3] introduced the logarithmic function by proving that

For any $a > 1$, there is a unique increasing continuous function g of the positive reals into the reals such that $g(xy) = g(x) + g(y)$ and $g(a) = 1$.

The approach adopted here leads to a new proof of the result of Dieudonné and will serve as an exercise on sequences for Calculus students, encouraging them to search for different viewpoints on well–established results.

In what follows, it will be shown that, for each $x \geq 0$, the sequence $(1 + \frac{x}{n})^n$ is monotonic and bounded and, hence, converges. The existence of the limit is extended to all real numbers x, by showing that

$$\lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^n = \frac{1}{\lim_{n\to\infty}(1 + \frac{x}{n})^n}.$$

Theorem 1. For each real number x, $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$ exists.

Proof. First it will be established that $\left(1 + \frac{x}{n}\right)^n$ is nondecreasing and bounded above for each nonnegative x. The proof is similar to the proof in [1] that $\left(1 + \frac{1}{n}\right)^n$ is increasing and bounded above.

Proof that $\left(1 + \frac{x}{n}\right)^n$ is nondecreasing. Let $x \geq 0$ and $a_n = \left(1 + \frac{x}{n}\right)^n$. Then

$$a_{n+1} - a_n = \left(1 + \frac{x}{n+1}\right)^{n+1} - \left(1 + \frac{x}{n}\right)^n$$

$$= \left(1 + \frac{x}{n+1}\right)^{n+1} - \left(1 + \frac{x}{n}\right)^{n+1} + \left(1 + \frac{x}{n}\right)^{n+1} - \left(1 + \frac{x}{n}\right)^n.$$

It is seen from (1) that

$$\left(1 + \frac{x}{n+1}\right)^{n+1} - \left(1 + \frac{x}{n}\right)^{n+1} = \frac{-x}{n(n+1)}\sum_{k=0}^{n}\left(1 + \frac{x}{n+1}\right)^k\left(1 + \frac{x}{n}\right)^{n-k}$$

$$\geq \frac{-x}{n(n+1)}\sum_{k=0}^{n}\left(1 + \frac{x}{n}\right)^n = \frac{-x}{n(n+1)}(n+1)\left(1 + \frac{x}{n}\right)^n = \frac{-x}{n}\left(1 + \frac{x}{n}\right)^n,$$

and clearly,

$$\left(1+\frac{x}{n}\right)^{n+1}-\left(1+\frac{x}{n}\right)^{n}=\left(1+\frac{x}{n}\right)^{n}\left(1+\frac{x}{n}-1\right)=\left(1+\frac{x}{n}\right)^{n}\frac{x}{n}.$$

Therefore,

$$a_{n+1}-a_n \geq \frac{-x}{n}\left(1+\frac{x}{n}\right)^{n}+\frac{x}{n}\left(1+\frac{x}{n}\right)^{n}=0.$$

The proof shows that for positive x, the sequence a_n is strictly increasing.

Proof that a_n is bounded. Consider the difference $\left(1+\frac{x}{mn}\right)^{n}-1$, where n and m are positive integers with $m>x$. From (1)

$$\begin{aligned}\left(1+\frac{x}{mn}\right)^{n}-1&=\frac{x}{mn}\sum_{k=0}^{n-1}\left(1+\frac{x}{mn}\right)^{k}\leq\frac{x}{mn}\sum_{k=0}^{n-1}\left(1+\frac{x}{mn}\right)^{n}\\&=\frac{x}{mn}n\left(1+\frac{x}{mn}\right)^{n}=\frac{x}{m}\left(1+\frac{x}{mn}\right)^{n}.\end{aligned}$$

Thus, for positive integers n,

$$\left(1+\frac{x}{mn}\right)^{n}-\frac{x}{m}\left(1+\frac{x}{mn}\right)^{n}=\left(1+\frac{x}{mn}\right)^{n}\left(1-\frac{x}{m}\right)\leq 1.$$

Hence, $\left(1+\frac{x}{mn}\right)^{mn}\left(1-\frac{x}{m}\right)^{m}\leq 1$. Since $\left(1+\frac{x}{n}\right)^{n}$ is nondecreasing and $mn\geq n$, we have

$$\left(1+\frac{x}{n}\right)^{n}\left(1-\frac{x}{m}\right)^{m}\leq 1 \quad\text{and so}\quad \left(1+\frac{x}{n}\right)^{n}\leq\left(\frac{m}{m-x}\right)^{m}.$$

Therefore, a_n is bounded. Theorem 1 is completed by employing (1) to show the following.

$\underline{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\left(1-\frac{x}{n}\right)^n = 1.}$

$$0 \le 1-\left(1-\frac{x^2}{n^2}\right)^n = \frac{x^2}{n^2}\sum_{k=0}^{n-1}\left(1-\frac{x^2}{n^2}\right)^k \le \frac{x^2}{n^2}\sum_{k=0}^{n-1}1 = \frac{x^2}{n} \to 0.$$

From Theorem 1 it follows that f defined by $f(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ is a real–valued function with the collection of reals as its domain. In the sequel, f will be this function. In Theorem 2 the identity in (1) is applied to produce some properties of f, including an elegant proof that $f' = f$ where f' represents the derivative function for f.

Theorem 2. The function f is a strictly increasing, continuous, differentiable function onto the positive reals satisfying

(i) $f(x+y) = f(x)f(y)$,
(ii) $f(rx) = (f(x))^r$ for each rational r, and
(iii) $f' = f$.

Proof that f is continuous. For real numbers x and a satisfying $|x-a| < 1$,

$$|f(x)-f(a)| \le \lim_{n\to\infty}\frac{|x-a|}{n}\sum_{k=0}^{n-1}\left(1+\frac{|x|}{n}\right)^k\left(1+\frac{|a|}{n}\right)^{n-1-k} \le |x-a|f(1+|a|).$$

Therefore, $\lim_{x\to a} f(x) = f(a)$.

Proof that $f' = f$. For any $x, a,\ x \ne a$,

$$\frac{\left(1+\frac{x}{n}\right)^n - \left(1+\frac{a}{n}\right)^n}{x-a} = \frac{1}{n}\sum_{k=0}^{n-1}\left(1+\frac{x}{n}\right)^k\left(1+\frac{a}{n}\right)^{n-1-k}.$$

So for any nonnegative $x, a,\ x \ne a$

$$\left(1+\frac{\min\{x,a\}}{n}\right)^{n-1} < \frac{\left(1+\frac{x}{n}\right)^n - \left(1+\frac{a}{n}\right)^n}{x-a} < \left(1+\frac{\max\{x,a\}}{n}\right)^{n-1}.$$

Letting $n \to \infty$,

$$f(\min\{x,a\}) \le \frac{f(x)-f(a)}{x-a} \le f(\max\{x,a\}). \tag{$*$}$$

For any nonpositive $x, a,\ x \neq a$

$$\frac{f(x)-f(a)}{x-a} = \frac{f(-x)-f(-a)}{f(-x)f(-a)(-x-(-a))}$$

and from inequality (*)

$$\frac{f(\min\{-x,-a\})}{f(-x)f(-a)} \le \frac{f(-x)-f(-a)}{f(-x)f(-a)(-x-(-a))} \le \frac{f(\max\{-x,-a\})}{f(-x)f(-a)}$$

$$\frac{f(\min\{-x,-a\})}{f(-x)f(-a)} \le \frac{f(x)-f(a)}{x-a} \le \frac{f(\max\{-x,-a\})}{f(-x)f(-a)}. \tag{$**$}$$

It follows from inequalities (*), (**), continuity of the functions $f, \max, \min$, and the "squeezing principle" that

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f(a).$$

<u>Proof of (i)</u>. For real numbers x and y

$$|f(x+y)-f(x)f(y)| \le \lim_{n\to\infty} \frac{|xy|}{n^2} \sum_{k=0}^{n-1} \left(1+\frac{|x+y|}{n}\right)^k \left(1+\frac{|x+y|}{n}+\frac{|xy|}{n^2}\right)^{n-1-k}$$

$$\le \lim_{n\to\infty} \frac{|xy|}{n} f(|x+y|+|xy|) = 0.$$

Proof of (ii). From (i) and induction, it follows that $f(mx) = (f(x))^m$ for all nonnegative integers m and real x. The identity $f(x)f(-x) = 1$ may then be used to prove $f(mx) = (f(x))^m$ for all integers m and real x. It is now clear that

$$f(1) = f\left(n\left(\frac{1}{n}\right)\right) = \left(f\left(\frac{1}{n}\right)\right)^n \text{ and } (f(1))^{\frac{1}{n}} = f\left(\frac{1}{n}\right)$$

for each positive integer n and hence, $f(r) = (f(1))^r$ for each rational r.

The function f is onto the set of positive reals. Let $z > 0$ and choose an integer m such that $m > z + \frac{1}{z}$. Since $f(1) > 2$, it follows that $f(m) = (f(1))^m > 2^m > m > z + \frac{1}{z} > z$ and that

$$f(-m) = \frac{1}{f(m)} < \frac{1}{z + \frac{1}{z}} < z.$$

By the Intermediate Value Theorem there is an x such that $f(x) = z$.

The function f is strictly increasing. This is a consequence of the facts that $f = f'$ and that f has positive values. It is instructive to see a proof using (1). If x and y are nonnegative and $x < y$ then

$$\begin{aligned} f(y) - f(x) &= \lim_{n\to\infty} \left[\left(1 + \frac{y}{n}\right)^n - \left(1 + \frac{x}{n}\right)^n\right] \\ &= \lim_{n\to\infty} \frac{y-x}{n} \sum_{k=1}^{n-1} \left(1 + \frac{y}{n}\right)^k \left(1 + \frac{x}{n}\right)^{n-1-k} \geq (y-x)f(x) > 0. \end{aligned}$$

If x and y are nonpositive and $x < y$ then $-x$ and $-y$ are nonnegative and $-y < -x$. Hence, $f(-y) < f(-x)$ and $f(x)f(y)f(-y) < f(-x)f(x)f(y)$, so $f(x) < f(y)$. Finally, if $x < y$ and $0 \in [x, y]$, then $f(y) - f(x) = (f(y) - f(0)) + (f(0) - f(x)) > 0$.

Theorem 3. If g is a continuous real–valued function on the reals satisfying

(i) $g(x + y) = g(x)g(y)$ and
(ii) $g(1) = f(1)$,

then $g = f$.

Proof. It will be sufficient to show that $g(r) = f(r)$ for rational r. From (i) and (ii), $f(1) = g(1) = g(1+0) = g(0)g(1) = g(0)f(1)$, so $g(0) = 1$. Hence, $g(x)g(-x) = g(0) = 1$. These properties of g and arguments like those in the proof of Theorem 2 (ii) will establish that $g(r) = (g(1))^r = (f(1))^r = f(r)$ for rational r.

Remark 1. The function f^{-1} is a continuous strictly increasing function from the positive reals onto the reals satisfying $f^{-1}(xy) = f^{-1}(x) + f^{-1}(y)$, $f^{-1}(1) = 0$, and $(f^{-1})'(x) = 1/x$. The function f^{-1} is of course customarily called the logarithm function.

Remark 2. For $a > 0$ and real x, a^x may now be defined as $f(xf^{-1}(a))$.

The final results in this article illustrate an interesting method of proof. Another property of f is offered in Theorem 4.

Theorem 4. For any $z > 0$, some integer m satisfies $f(m) \leq z < f(m+1)$.

Proof. From above, there is an integer n such that $f(n) \leq z$. Let $\mathcal{A}$ be the collection of such $f(n)$ and let $p = \sup \mathcal{A}$. Since $f(1) > 1$ it follows that $p/f(1) < p$. Choose an integer m satisfying $f(m) \leq p$, $p/f(1) < f(m)$, and consequently $p < f(m+1)$. Since $m+1$ is an integer and $f(m+1) \notin \mathcal{A}$, m satisfies $f(m) \leq z < f(m+1)$.

Remark 3. It is interesting that an argument similar to that used in the proof of Theorem 4 produces the following simple proof that between any two distinct reals x and y there is a rational, although it is not as geometrical in nature as the usual proof. (The essence of the technique usually employed is to show that there is an interval $I = [a, b]$ with integer endpoints such that $x,\ y \in I$ and then to partition such an interval I into n subintervals of equal length, where $|x - y| > (b - a)/n$). Suppose $x < y$, Q is the set of rationals, and let $S = \{r \in Q : r < y\}$. Then $S \neq \emptyset$ (the set of integers has no lower bound) and y is an upper bound for S. If $s = \sup S$ then for each positive integer m there is an $r_m \in S$ satisfying $s < r_m + 1/m$. Then $r_m + 1/m \notin S$, $r_m + 1/m \in Q$ and consequently, for such m,

$$\begin{cases} r_m \leq s < r_m + 1/m \\ r_m < y \leq r_m + 1/m. \end{cases} \tag{$***$}$$

From $(***)$, $0 \leq y - s < 1/m$ for each positive integer m and hence, $s = y$. Since $x < y$ there is an $r \in S$ such that $x < r$.

References

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