SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

141. [2003, 200] *Proposed by Kenneth B. Davenport, Frackville, Pennsylvania.*

(a) Show that

$$\sum_{k=1}^{n-1} \frac{1}{1-\tan^2\left(\frac{k\pi}{2n}\right)} = \frac{n-1}{2},$$

for $n = 3, 5, 7, 9, \ldots$.

(b) Show that

$$\sum_{k=1}^{n-1} \frac{1}{1+\tan^2\left(\frac{k\pi}{2n}\right)} = \frac{n-1}{2},$$

for $n = 2, 3, 4, 5, \ldots$.

Solution by Joe Howard, Portales, New Mexico. Note that $\cos\theta = -\cos(\pi-\theta)$ and $\sec\theta = -\sec(\pi-\theta)$. By pairing in this way (Ex: $\cos\frac{\pi}{3} + \cos\frac{2\pi}{3} = 0$) it follows that

$$\sum_{k=1}^{n-1} \cos\frac{k\pi}{n} = 0 \text{ and } \sum_{k=1}^{n-1} \sec\frac{k\pi}{n} = 0 \text{ for } n = 3, 5, 7, 9, \ldots .$$

Also, $\cos\frac{\pi}{2} = 0$ so

$$\sum_{k=1}^{n-1} \cos\frac{k\pi}{n} = 0 \text{ for } n = 2, 4, 6, 8, \ldots .$$

(a) For $n = 3, 5, 7, 9, \ldots$

$$\sum_{k=1}^{n-1} \frac{1}{1-\tan^2\frac{k\pi}{2n}} = \sum_{k=1}^{n-1} \frac{\cos^2\frac{k\pi}{2n}}{\cos^2\frac{k\pi}{2n} - \sin^2\frac{k\pi}{2n}} = \frac{1}{2}\sum_{k=1}^{n-1} \frac{1+\cos\frac{k\pi}{n}}{\cos\frac{k\pi}{n}}$$

$$= \frac{1}{2}\sum_{k=1}^{n-1}\left(1+\sec\frac{k\pi}{n}\right) = \frac{n-1}{2} + \frac{1}{2}\sum_{k=1}^{n-1}\sec\frac{k\pi}{n} = \frac{n-1}{2}.$$

(b) For $n = 2, 3, 4, 5, \ldots$

$$\sum_{k=1}^{n-1} \frac{1}{1+\tan^2\frac{k\pi}{2n}} = \sum_{k=1}^{n-1}\cos^2\frac{k\pi}{2n} = \frac{1}{2}\sum_{k=1}^{n-1}\left(1+\cos\frac{k\pi}{n}\right)$$

$$= \frac{n-1}{2} + \frac{1}{2}\sum_{k=1}^{n-1}\cos\frac{k\pi}{n} = \frac{n-1}{2}.$$

Also solved by Joe Flowers, Texas Lutheran University, Sequin, Texas; Don Redmond, Southern Illinois University, Carbondale, Illinois; Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri; Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan; and the proposer.

Comment by the proposer. For a related problem, see Problem H-566 [2000, 377; 2001, 474–476] in *The Fibonacci Quarterly.*

142. [2003, 201] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Solve the differential equation

$$y' + \frac{y}{x^x}(x^x \ln y)^n + y(\ln x \ln y) = 0,$$

where n is any real number.

Solution by Joe Flowers, Texas Lutheran University, Sequin, Texas. The substitution $v = x^x \ln y$ leads in straightforward fashion to the Bernoulli equation

$$v' - v = -v^n.$$

If $n = 1$, then $v' = 0$, so $v = c$, hence

$$y = e^{\frac{c}{x^x}}.$$

For $n \neq 1$, the substitution $v = w^{\frac{1}{1-n}}$ yields the linear equation

$$w' + (n-1)w = n - 1.$$

Applying the integrating factor $e^{(n-1)x}$, we obtain the solution

$$w = 1 + ce^{(1-n)x}$$

which then gives

$$v = \left(1 + ce^{(1-n)x}\right)^{\frac{1}{1-n}}$$

and finally

$$y = \exp\left(\frac{(1 + ce^{(1-n)x})^{\frac{1}{1-n}}}{x^x}\right).$$

Also solved by Kenneth B. Davenport, Frackville, Pennsylvania, Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan, J. D. Chow, Edinburg, Texas; and the proposer.

144. [2003, 201] *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan.*

Prove that in any triangle the following inequality holds:

$$\sum \frac{b+c}{a} \tan\frac{B}{2} \tan\frac{C}{2} \geq 2,$$

where the notations are usual.

Solution I by Mangho Ahuja, Southeast Missouri State University, Cape Girardeau, Missouri. Using the identity

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

where $s = (a+b+c)/2$, we have

$$\tan\frac{B}{2} \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \sqrt{\frac{(s-b)(s-a)}{s(s-c)}} = \frac{s-a}{s}.$$

The given expression

$$\begin{aligned}
\sum \frac{b+c}{a} \tan\frac{B}{2} \tan\frac{C}{2} &= \sum \frac{b+c}{a} \cdot \frac{s-a}{s} = \sum \frac{b+c}{a}\left(1 - \frac{a}{s}\right) \\
&= \sum \frac{b+c}{a} - \sum \frac{b+c}{s} = \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} - \frac{2a+2b+2c}{s} \\
&= \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) - \frac{4s}{s} \\
&= \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) - 4.
\end{aligned}$$

But using the AM-GM inequality, the quantity

$$\frac{a}{b} + \frac{b}{a} \geq 2.$$

Hence, the given expression is greater than or equal to

$$2 + 2 + 2 - 4 = 2.$$

Solution II by Joe Howard, Portales, New Mexico. We use the formula that

$$\sum_{\text{cyclic}} \tan \frac{B}{2} \tan \frac{C}{2} = 1,$$

the inequality

$$u + \frac{1}{u} \geq 2,$$

and Chebyshev's Inequality

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right),$$

where $x_1 \geq x_2 \geq \ldots \geq x_n > 0$ and $y_1 \geq y_2 \geq \ldots \geq y_n > 0$. Without loss of generality assume $a \geq b \geq c$. Then

$$\frac{a+b}{c} \geq \frac{a+c}{b} \geq \frac{b+c}{a}$$

and

$$\tan \frac{A}{2} \tan \frac{B}{2} \geq \tan \frac{A}{2} \tan \frac{C}{2} \geq \tan \frac{B}{2} \tan \frac{C}{2}.$$

Then

$$\sum_{\text{cyclic}} \frac{b+c}{a} \tan\frac{B}{2} \tan\frac{C}{2}$$

$$\geq \frac{1}{3}\left(\frac{b}{a}+\frac{c}{a}+\frac{a}{b}+\frac{c}{b}+\frac{a}{c}+\frac{b}{c}\right)\left(\sum_{\text{cyclic}} \tan\frac{B}{2}\tan\frac{C}{2}\right)$$

$$\geq \frac{1}{3}(6)(1) = 2.$$

Also solved by the proposer.