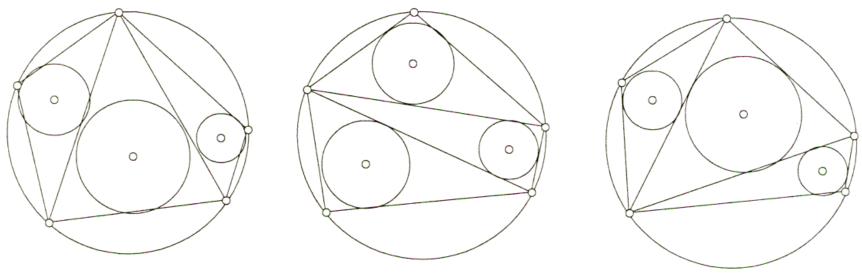
JAPANESE THEOREM: A LITTLE KNOWN THEOREM WITH MANY PROOFS – PART I

Mangho Ahuja, Wataru Uegaki, and Kayo Matsushita

[Note: (A), (U), (M) refer to the authors' names. The pronoun I always means (A).]

Japanese Theorem. Triangulate a cyclic polygon by lines drawn from any vertex. The sum of the radii of the incircles of the triangles is independent of the vertex chosen.



1. Background. I (A) found this theorem in a 1993 article [5], where the author Nick Mackinnon wrote " ... I have used the above theorem as a starter for course work, not expecting a proof of the theorem (I can't even prove it myself) ..." It was the author's remark in parenthesis that intrigued me. In fall 1995 I assigned this theorem as a problem to Cathy, a Masters Degree student. By spring 1996 both Cathy and I had a proof of the theorem [3]. But one question remained: Why is it called the Japanese Theorem? My long search for the answer ended when I received a 15–page fax in English, French, and Japanese from Professor Yoshida of Kyoto University.

The theorem (Quadrilateral case) had originated in China [6]. So, when it came to Japan around 1900, it was known as the "Chinese Theorem" [4]. Later, when Y. Mikami generalized it from a quadrilateral to a polygon, the name remained the "Chinese Theorem". So, who coined the term "The Japanese Theorem"? This theorem, without a name, appeared in a 1906 article entitled, "Japanese Mathematics" [2]. We believe this led the later authors to call it the "Japanese Theorem".

This theorem is displayed on a wooden tablet in a Shinto shrine. The hanging of such tablets showing mathematical theorems was a common custom in Edo era in Japan [1]. These tablets, called *Sangaku* in Japanese, can be seen all over Japan. The Sangaku of our theorem was once hanging in the Tsuruoka–San'nosha shrine in the Ushu area (at present Yamagata and Akita prefectures of Japan), but it has since disappeared.

To learn more about the history of the theorem, in summer 1999 I traveled to Japan, where (M) and I searched the libraries of Kyoto University for references. At the same time Professor (U) of Mie University was examining and analyzing every available document relevant to the theorem. His findings are published in a paper (in Japanese) in the *Journal of Mie University* [7]. The authors plan to write a detailed history of the theorem in another paper, leaving the present paper to focus on the geometry only.

2. Plan. In part I, we state a few elementary results, E1, E2, ..., E5, and then derive results G1, G2, ..., G8 in Geometry, of which a few are well–known. We end part I with the oldest proof of the theorem. In part II we show a variety of proofs – five different proofs by Japanese mathematicians. We end our paper with a discussion of the generalized (polygonal) case of the theorem and the conclusion.

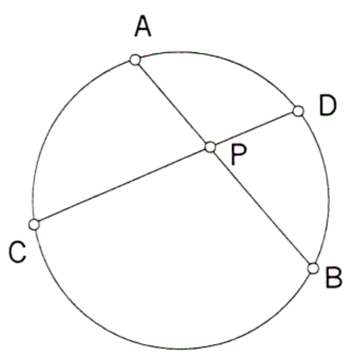
Elementary Results. In any triangle ABC, the following are true:

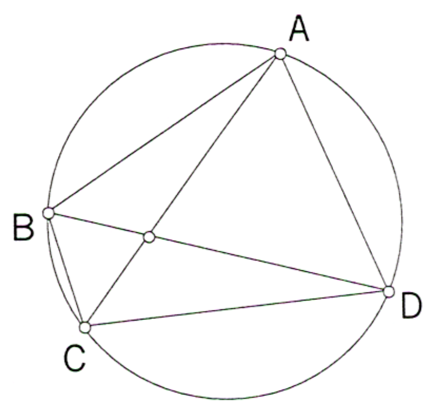
(E1) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(E2) $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$.

(E3) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

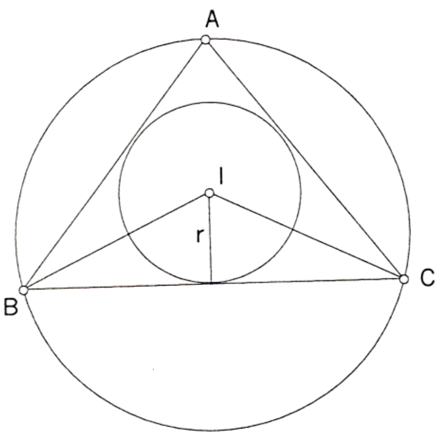
(E4) If chords AB and CD of a circle intersect at P, then $PA \cdot PB = PC \cdot PD$.





(E5) Ptolemy's Theorem. If a quadrilateral $ABCD$ is inscribed in a circle, then $AB \cdot CD + BC \cdot AD = AC \cdot BD$.

3. Results from Plane Geometry. We will prove results G1 to G8 (some of these are known theorems) needed for the proofs that follow. For a triangle ABC, let O, R, and I, r denote the center and radius of the circumcircle and the incircle, respectively.



(G1) $r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$.

Proof. Now

$$a = BC = r\cot\frac{B}{2} + r\cot\frac{C}{2} = r\left[\frac{\cos\frac{B}{2}\sin\frac{C}{2} + \cos\frac{C}{2}\sin\frac{B}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\right]$$

$$= r\left[\frac{\sin\left(\frac{B}{2}+\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}\right] = r\left[\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\right].$$

Thus,

$$r = a\left[\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}\right].$$

But

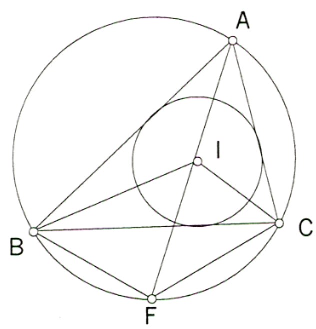
$$a = 2R \sin A = 4R \sin \frac{A}{2} \cos \frac{A}{2},$$

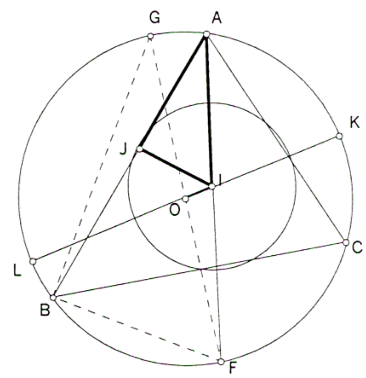
hence,

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

(G2) If AI produced meets the circumcircle in F, then FB, FC, FI are equal. In other words the points B, C, and I lie on a circle with center F.

Proof. Clearly, $\angle BIF = \angle IBA + \angle BAI = B/2 + A/2$. Also, $\angle IBF = \angle IBC + \angle CBF = \angle IBC + \angle CAF = B/2 + A/2$. Thus, $\angle BIF = \angle IBF$ and $FI = FB$. Similarly, $FI = FC$. So the points B, I, and C lie on a circle with center F.

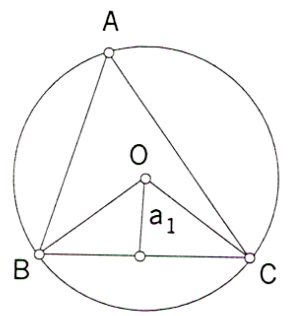


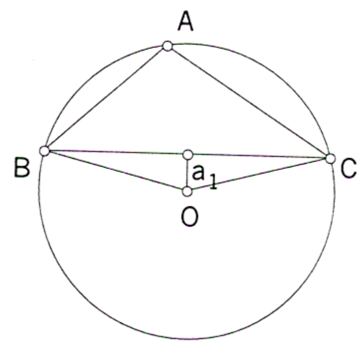


(G3) $R^2 - 2Rr = OI^2$ (This result is also known as Chapple's Theorem.)

Proof. Extend AI till it meets the circumcircle in F. We draw two diameters FOG and $KIOL$ through O. Let IJ be the perpendicular from I to side AB, then $IJ = r$. Since $\angle BAF = \angle BGF$, the two right triangles AJI and GBF are similar. Hence, $r/AI = BF/2R$, or $2Rr = AI \cdot BF = AI \cdot IF$, which by (E4) equals $LI \cdot IK = (R + OI)(R - OI) = R^2 - OI^2$. Hence, $R^2 - 2Rr = OI^2$.

(G4) (Carnot's Theorem) Let a_1, b_1, and c_1 denote the lengths of perpendiculars from O to sides BC, CA, and AB, respectively. Then $R + r = b_1 + c_1 + a_1$, if O lies within the triangle ABC, and $R + r = b_1 + c_1 - a_1$ if say OA intersects BC.

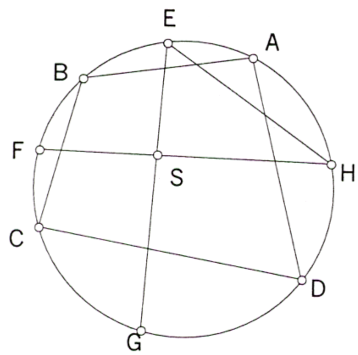


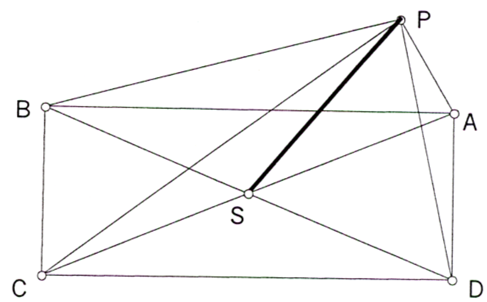


Proof. Since $\angle BOC = 2\angle A$, the length a_1 is either $R\cos A$, or $R\cos(\pi - A) = -R\cos A$, depending upon the position of O. Thus, $b_1 + c_1 \pm a_1 = R[\cos B + \cos C + \cos A]$ which equals $R[1 + 4\sin A/2 \sin B/2 \sin C/2]$ by (E3), and equals $R + r$ by (G1). Thus, $R + r = b_1 + c_1 \pm a_1$.

(G5) Let $ABCD$ be a quadrilateral inscribed in a circle. If E, F, G, and H denote the midpoints of the arcs AB, BC, CD, and DA, respectively, then the lines EG and FH are perpendicular.

Proof. If we suppose that the arcs AB, BC, CD, and DA make angles 2α, 2β, 2γ, and 2δ at the center of the circle, then the sum $2\alpha + 2\beta + 2\gamma + 2\delta = 2\pi$. Now the arc EF makes angle $\alpha + \beta$ at the center, so $\angle EHF = (1/2)(\alpha + \beta)$. Similarly, arc GH makes angle $\gamma + \delta$ at the center, hence, $\angle GEH = (1/2)(\gamma + \delta)$. If the lines EG and FH intersect in S, then the sum of the two angles SHE and HES of triangle ESH is $(1/2)(\alpha + \beta + \gamma + \delta) = (1/2)\pi$, making the third angle $\angle ESH$ a right angle.

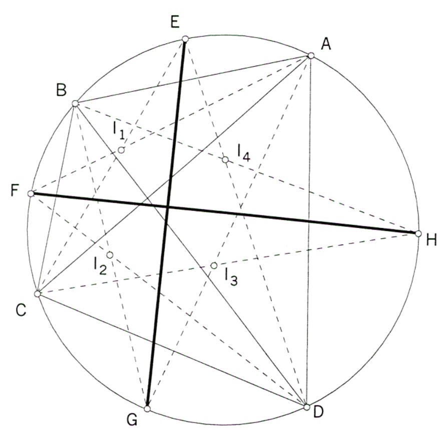




(G6) Let $ABCD$ be a rectangle whose diagonals intersect at S. Let P be any point in a plane. Then $PA^2 + PC^2 = PB^2 + PD^2 = 2PS^2 + 2AS^2$.

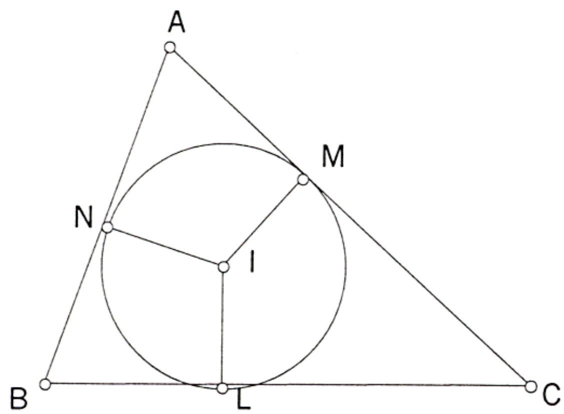
Proof. Since PS is the median for triangles PCA and PBD, we have $PA^2 + PC^2 = 2AS^2 + 2PS^2 = PB^2 + PD^2$.

(G7) If I_1, I_2, I_3, I_4 are the incenters of triangles ABC, BCD, CDA, and DAB, respectively, then the figure $I_1I_2I_3I_4$ is a rectangle.



Proof. Let E, F, G, H denote the midpoints of the arcs AB, BC, CD, and DA, respectively. Then E lies on the bisectors of angles BCA and BDA. Similarly, each of the points F, G, and H lies on two angle bisectors as shown in the diagram. This means that the bisectors AF and CE intersect at I_1, and the same for I_2, I_3, and I_4. The equal arcs AH and HD make equal angles at F, hence, FH is the bisector of angle AFD. For the same reason, FH is also the angle bisector of angle BHC. By using (G2) on triangles ABC and BDC, we get $FI_1 = FB = FC = FI_2$. This makes I_1I_2 perpendicular to the bisector of angle I_1FI_2, which is FH. Similarly, I_3I_4 is perpendicular to FH, and this makes I_1I_2 and I_3I_4 parallel lines. Similarly, I_1I_4 and I_2I_3 are parallel lines, both being perpendicular to EG. But (G5) says that EG and FH are perpendicular. Hence, the figure $I_1I_2I_3I_4$ is a rectangle.

(G8) Let IL, IM, IN be the perpendiculars from the incenter I to the sides BC, CA, and AB of triangle ABC. Then, $AM = AN = (1/2)(b + c - a)$, $BL = BN = (1/2)(c + a - b)$, and $CL = CM = (1/2)(a + b - c)$.



Proof. Note that $AM = AN$, $BL = BN$, $CL = CM$. Thus, $2AM+2BL+2CL$ equals $a + b + c$, or $2AM + 2a = a + b + c$. Hence, $AM = (1/2)(b + c - a)$.

4. Earliest Attempt. Japan was a closed society until 1854, when Commodore Perry forced open its doors, and Japan began to exchange goods and knowledge with the western countries. The native Japanese mathematics prior to its contact with the European world was known as *Wasan,* which means Japanese Mathematics. The earliest proof of the Japanese Theorem is found in a book on *Wasan* written by Tameyuki Yoshida [9]. We are not sure if Yoshida himself gave this proof. The interesting thing about this proof, besides being the earliest, is that it is based on just two properties of the circle – (1) from any point outside the circle, tangents drawn to the circle have equal length, and (2) an arc of a circle makes the same angles at any point on the circle. We now present this proof.

Japanese Theorem (Quadrilateral Case). Let $ABCD$ be a quadrilateral inscribed in a circle. Let r_1, r_2, r_3, r_4 be the radii of the circles C_1, C_2, C_3, and C_4 inscribed in triangles ABC, BCD, CDA, and DAB, respectively. Then $r_1 + r_3 = r_2 + r_4$.

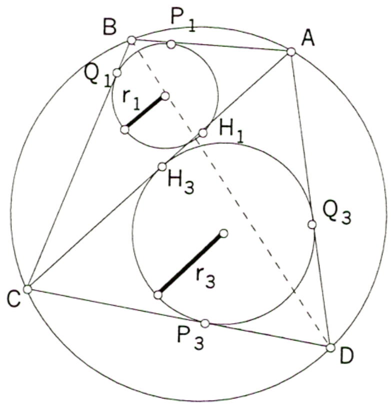


Diagram (A)

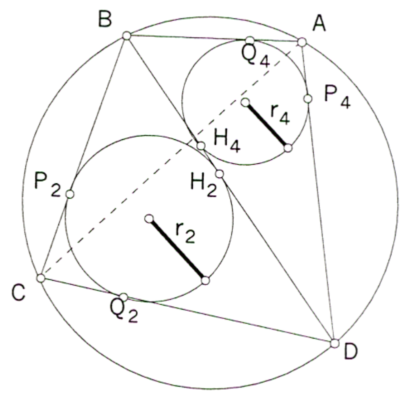


Diagram (B)

Proof. Let P_i, Q_i, and H_i denote the points where the circle C_i touches the sides as shown in diagrams (A) and (B). The first step of the proof is to show that H_1H_3 and H_2H_4 are equal. To do so, let $\mathbf{E}$ denote the expression $AD + BC - AB - CD$. From diagram (A), $\mathbf{E} = (AQ_3 + DQ_3) + (BQ_1 + CQ_1) - (AP_1 + BP_1) - (CP_3 + DP_3)$. But $AP_1 = AH_1$, $AQ_3 = AH_3$, $CQ_1 = CH_1$, $CP_3 = CH_3$, and so on. Hence, $\mathbf{E} = AH_3 + CH_1 - AH_1 - CH_3 = (AH_3 - AH_1) + (CH_1 - CH_3) = 2H_1H_3$. From diagram (B), $\mathbf{E} = (AP_4 + DP_4) + (BP_2 + CP_2) - (AQ_4 + BQ_4) - (CQ_2 + DQ_2)$. But $BP_2 = BH_2$, $BQ_4 = BH_4$, and so on. Thus, $\mathbf{E} = DH_4 + BH_2 - BH_4 - DH_2 = (DH_4 - DH_2) + (BH_2 - BH_4) = 2H_2H_4$. Hence, $H_1H_3 = H_2H_4$.

Let arcs AB, BC, CD, DA make angles 2α, 2β, 2γ, and 2δ, respectively at points on the circle. This means $\angle ACB = \angle ADB = 2\alpha$, $\angle BAC = \angle BDC = 2\beta$,

$\angle CAD = \angle CBD = 2\gamma$, and $\angle DBA = \angle DCA = 2\delta$. Then

$$\tan\alpha = \frac{r_1}{CH_1} = \frac{r_4}{DH_4}, \quad \tan\beta = \frac{r_1}{AH_1} = \frac{r_2}{DH_2},$$

$$\tan\gamma = \frac{r_3}{AH_3} = \frac{r_2}{BH_2}, \quad \tan\delta = \frac{r_3}{CH_3} = \frac{r_4}{BH_4}.$$

This gives us $r_1DH_4 - r_4CH_1 = 0$, $r_2AH_1 - r_1DH_2 = 0$, $r_3BH_2 - r_2AH_3 = 0$, and $r_4CH_3 - r_3BH_4 = 0$. On adding these, we get $r_1(DH_4 - DH_2) + r_3(BH_2 - BH_4) = r_2(AH_3 - AH_1) + r_4(CH_1 - CH_3)$, or $r_1(H_2H_4) + r_3(H_2H_4) = r_2(H_1H_3) + r_4(H_1H_3)$. But $H_2H_4 = H_1H_3$, hence, $r_1 + r_3 = r_2 + r_4$. This completes the proof.

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Mangho Ahuja
2575 Fairlane
Cape Girardeau, MO 63701
email: mangho@showme.net

Wataru Uegaki
Department of Math Education
Mie University
1515 Kamihama–cho
Tsu–shi, MIE
Japan 514–8507
email: uegaki@edu.mie-u.ac.jp

Kayo Matsushita
Center for the Promotion of Excellence in Higher Education
Kyoto University
Kyoto 606-8501
Japan
email: kmatsu@hedu.mbox.media.kyoto-u.ac.jp