GENERALIZED HADAMARD PRODUCT SETS OF POLYNOMIALS IN SIMPLE SETS

K. A. M. Sayyed, M. S. Metwally, and M. T. Mohamed

Abstract. In this paper a new basic set of polynomials of one complex variable is defined and some of its properties are studied. Its representation of regular functions under certain conditions due to Whittaker, Boas, and Newns in certain domains are investigated.

1. Introduction. A sequence $p_0(z), p_1(z), \ldots, p_n(z), \ldots$ of polynomials where

$$p_n(z) = \sum_k p_{n,k} z^k, \tag{1.1}$$

is said to form a basic set $\{p_n(z)\}$ in the sense of Whittaker or simply a W–basic set, if any polynomial, and in particular the monomial z^n, admits a unique finite representation of the form

$$z^n = \sum_k \pi_{n,k} p_k(z). \tag{1.2}$$

The row–finite matrices in (1.1) and (1.2) are denoted by $P = (p_{n,k})$ and $\Pi = (\pi_{n,k})$ and are called the matrix of coefficients and the matrix of operators of the set $\{p_n(z)\}$, respectively. Whittaker [8] has proved that a necessary and sufficient condition for the set $\{p_n(z)\}$ to be basic is that

$$P\Pi = \Pi P = I \tag{1.3}$$

where I is the infinite unit matrix, that is to say that the matrix of coefficients P admits a unique row finite inverse Π, i.e. $P^{-1} = \Pi$.

For example, the set $\{p_n(z)\}$ of polynomials in which the polynomial $p_n(z)$ is of degree n is necessarily basic and is called a simple set. Such a set for which

$$p_{n,n} = 1 \text{ for all } n \geq 0 \tag{1.4}$$

is called a simple monic set.

If the function $f(z)$ is regular at the origin, then it has a Taylor expansion in the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \ , \ a_n = \frac{f^{(n)}(0)}{n!}. \tag{1.5}$$

Substituting for z^n from (1.2) and rearranging the terms, (1.5) will be rewritten in the form

$$f(z) \sim \sum_{n=0}^{\infty} \Pi_n f(0) p_n(z), \tag{1.6}$$

where

$$\Pi_n f(0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \pi_{k,n}. \tag{1.7}$$

The series (1.6) is called a basic series associated with the function $f(z)$. If it converges uniformly to the function $f(z)$ in any circle $|z| \leq R$ in which the function is regular, then it is said that the set $\{p_n(z)\}$ represents $f(z)$ in $|z| \leq R$. The basic set $\{p_n(z)\}$ is effective in $|z| \leq R$ if it is represented in $|z| \leq R$ for every regular function $f(z)$.

Cannon [2] simplified the study of this subject by defining the Cannon sum $\omega_n(r)$ and the Cannon function $\omega(r)$, of the set $\{p_n(z)\}$, respectively, as follows

$$\omega_n(r) = \sum_k |\pi_{n,k}| M_k(r) \tag{1.8}$$

and

$$\omega(r) = \lim_{n\to\infty} \{\omega_n(r)\}^{\frac{1}{n}}, \tag{1.9}$$

where

$$M_n(r) = \max_{|z|=r} |p_n(z)|. \tag{1.10}$$

For effectiveness of the basic set $\{p_n(z)\}$ in the closed circle $|z| \leq r$ the necessary and sufficient condition is

$$\omega(r) = r, \ r > 0. \tag{1.11}$$

2. Generalized Hadamard Product Sets of Polynomials. Let $\{p_n(z)\}$ and $\{q_n(z)\}$ be two sets of polynomials of the single complex variable z, where $p_n(z) = \sum_j p_{n,j} z^j$ and $q_n(z) = \sum_j q_{n,j} z^j$. Define, for any two positive numbers μ and ν, the sets $\{p_n^\mu(z)\}$ and $\{q_n^\nu(z)\}$ such that the matrices of coefficients are $P^\mu = (p_{n,j}^\mu)$ and $Q^\nu = (q_{n,j}^\nu)$ whose elements are the μ–th and ν–th power of the elements of the matrices of P and Q, respectively; i.e. $p_n^\mu(z) = \sum_j p_{n,j}^\mu z^j$ and $q_n^\nu(z) = \sum_j q_{n,j}^\nu z^j$.

The set $\{U_n^{(\mu,\nu)}(z)\}$ of polynomials of the complex variable z in which

$$U_n^{(\mu,\nu)}(z) = \sum_j U_{n,j}^{(\mu,\nu)} z^j = \sum_{j=0}^{n-1} p_{n,j}^\mu q_{n,j}^\nu z^j + z^n$$

is called the Generalized Hadamard product set of polynomials of the single complex variable z of the two sets $\{p_n(z)\}$ and $\{q_n(z)\}$.

We note that the two original sets $\{p_n(z)\}$ and $\{q_n(z)\}$ may be two basic sets of polynomials and their generalized Hadamard product set may not be basic. This fact is illustrated by the following example.

Example 2.1. Suppose that the sets $\{p_n(z)\}$ and $\{q_n(z)\}$ are two basic sets of polynomials given by

$$p_n(z) = \begin{cases} z^{n+1} & ; \quad n \text{ even or zero} \\ 2z^{n-1} & ; \quad n \text{ odd} \end{cases}$$

and

$$q_n(z) = 3z^n \text{ for all } n = 0, 1, 2 \ldots.$$

Then the generalized Hadamard product set $\{U_n^{(2,3)}(z)\}$ will be such that

$$U_{n,k}^{(2,3)} = \begin{cases} p_{j,k}^2 q_{j,k}^3 = (p_{j,k}^2)(0) = 0 \ ; \ j \neq k \\ p_{j,k}^2 q_{j,k}^3 = (0)(3^3) = 0, \end{cases}$$

i.e., $U^{(2,3)}_{n,j} = 0$ for all n, j, and the set $\{U^{(2,3)}_n(z)\}$ cannot be basic.

Taking the two sets $\{p_n(z)\}$ and $\{q_n(z)\}$ to be simple sets, then the sets whose coefficients are the μ–th and ν–th power of the coefficients of these sets will also be simple sets, since this procedure does not affect the degree of the variable z, and their generalized Hadamard product set will also be simple. This means that the generalized Hadamard product set is a simple set whenever the two original sets are simple ones. In particular, the generalized Hadamard product set whose original sets are simple monic sets is a simple monic set also. In other words, since the original sets $\{p_n(z)\}$ and $\{q_n(z)\}$ are simple sets, then their matrices of coefficients P and Q are semi–lower matrices. So are the matrices $P^{\mu} = (p^{\mu}_{n,j})$ and $Q^{\nu} = (q^{\nu}_{n,j})$ and multiplying the corresponding elements of the last two matrices produces a semi–lower matrix. That is to say that the matrix $U^{(\mu,\nu)}$ of coefficients of the generalized Hadamard product set is a semi–lower matrix or the generalized Hadamard product set is a simple set which is necessarily basic, and the basic property of the generalized Hadamard product set of simple sets follows.

3. Effectiveness of Generalized Hadamard Product Sets of Polynomials In Simple Sets In Closed Circles. We begin the study of this property by showing that effectiveness of the two original sets in certain closed circles does not imply effectiveness of the generalized Hadamard product set in any closed circle. We give the following example, in which the set $\{p_n(z)\}$ was given by Nassif and Makar [6].

Example 3.1. Let $\{p_n(z)\}$ and $\{q_n(z)\}$ be simple monic sets of polynomials of the complex variable z, where

$$\begin{aligned} p_{2n}(z) &= z^{2n-1} + z^{2n} \\ p_{2n+1}(z) &= (2n+1)z^{2n-1} + (2n+1)z^{2n} + z^{2n+1};\ n \geq 1 \\ p_1(z) &= 1 + z \\ p_0(z) &= 1 \end{aligned}$$

and

$$\begin{aligned} q_{2n}(z) &= 2z^{2n-1} + z^{2n} \\ q_{2n+1}(z) &= 2z^{2n-1} + 2z^{2n} + z^{2n+1};\ n \geq 1 \\ q_1(z) &= 2 + z \\ q_0(z) &= 1. \end{aligned}$$

The set $\{p_n(z)\}$ is effective in $|z| \leq r$ for all r.

It is clear that $|q_{n,j}| \leq 2$; $n \geq 0$ and $0 \leq j \leq n-1$. Then by Cannon's Theorem [2] the set $\{q_n(z)\}$ is effective in $|z| \leq r$ for all $r \geq 3$.

Construct the generalized Hadamard product set $\{U_n^{(2,3)}(z)\}$ of polynomials as follows:

$$U_{2n}^{(2,3)}(z) = 2^3 z^{2n-1} + z^{2n},$$

$$U_{2n+1}^{(2,3)}(z) = 2^3(2n+1)^2 z^{2n-1} + 2^3(2n+1)^2 z^{2n} + z^{2n+1};\ n \geq 1,$$

$$U_1^{(2,3)}(z) = 2^3 + z$$

$$U_0^{(2,3)}(z) = 1.$$

Since the set $\{U_n^{(2,3)}(z)\}$ is a simple monic set, then it must be basic. Then according to Mursi and Makar [4] the necessary and sufficient condition for the set to be basic is that the row finite matrix of coefficients, $U^{(2,3)}$, must have a unique row finite inverse, $\bar{U}^{(2,3)}$, i.e. $U^{(2,3)}\bar{U}^{(2,3)} = \bar{U}^{(2,3)}U^{(2,3)} = I$. Then we have the following relations:

$$U_{n,m}^{(2,3)}\bar{U}_{n,m}^{(2,3)} = 1,$$

$$U_{2n,2n-1}^{(2,3)}\bar{U}_{2n-1,0}^{(2,3)} + U_{2n,2n}^{(2,3)}\bar{U}_{2n,0}^{(2,3)} = 0$$

and

$$U_{2n+1,2n-1}^{(2,3)}\bar{U}_{2n-1,0}^{(2,3)} + U_{2n+1,2n}^{(2,3)}\bar{U}_{2n,0}^{(2,3)} + U_{2n+1,2n+1}^{(2,3)}\bar{U}_{2n+1,0}^{(2,3)} = 0.$$

Substituting, we have

$$2^3\bar{U}_{2n-1,0}^{(2,3)}\bar{U}_{2n,0}^{(2,3)} = 0$$

and

$$2^3(2n+1)^2\bar{U}_{2n-1,0}^{(2,3)} + 2^3(2n+1)^2\bar{U}_{2n,0}^{(2,3)} + \bar{U}_{2n+1,0}^{(2,3)} = 0.$$

These equations give

$$\bar{U}_{2n+1,0}^{(2,3)} = 2^3(2n+1)^2(2^3-1)\bar{U}_{2n-1,0}^{(2,3)}. \tag{3.1}$$

Since

$$U_{0,0}^{(2,3)}\bar{U}_{0,0}^{(2,3)} + U_{0,1}^{(2,3)}\bar{U}_{1,0}^{(2,3)} + U_{0,2}^{(2,3)}\bar{U}_{2,0}^{(2,3)} + \cdots = 1$$

and

$$U_{1,0}^{(2,3)}\bar{U}_{0,0}^{(2,3)} + U_{1,1}^{(2,3)}\bar{U}_{1,0}^{(2,3)} + U_{1,2}^{(2,3)}\bar{U}_{2,0}^{(2,3)} + \cdots = 0.$$

From which we see that

$$\bar{U}_{1,0}^{(2,3)} = -(2)^3.$$

It follows from relation (3.1) that

$$\bar{U}_{3,0}^{(2,3)} = 2^3(3)^2(2^3-1)\bar{U}_{1,0}^{(2,3)} = -(2^3)(2^3-1)(3),$$

$$\bar{U}_{5,0}^{(2,3)} = 2^3(5)^2(2^3-1)\bar{U}_{3,0}^{(2,3)} = -(2^3)^3(2^3-1)^2(5)^2(3)^2$$

and

$$\bar{U}_{7,0}^{(2,3)} = -(2^3)^4(2^3-1)^3(7)^2(5)^2(3)^2.$$

From this recurrence relation we get

$$\begin{aligned}\bar{U}_{2n+1,0}^{(2,3)} &= -(2^3)^{n+1}(2^3-1)^n(2n+1)^2(2n-1)^2(2n-3)^2\dots 1\\ &= -(8)^{n+1}(7)^n[(2n+1)(2n-1)(2n-3)\dots 1]^2.\end{aligned}$$

Hence,

$$\begin{aligned}\omega_{2n+1}^{(2,3)}(r) &= \max_{|z|=r}\sum_j |\bar{U}_{2n+1,j}^{(2,3)}||U_j^{(2,3)}(z)|\\ &\geq |\bar{U}_{2n+1,0}^{(2,3)}|\max_{|z|=r}|U_0^{(2,3)}(z)|\\ &= (8)^{n+1}(7)^n[(2n+1)(2n-1)\dots 1]^2 > n^{2n}.\end{aligned}$$

Therefore, $\omega^{(2,3)}(r) = +\infty$ for all r and the generalized Hadamard product set cannot be effective in any closed circle. Take for example $r = 10$, then all the sets $\{p_n(z)\}$, $\{q_n(z)\}$, $\{p_n^{\mu}(z)\}$ and $\{q_n^{\nu}(z)\}$, are simple monic sets effective in $|z| \leq r$, for $r = 10$, but the generalized Hadamard product set $\{U_n^{(2,3)}(z)\}$ is not effective. So effectiveness of the two original sets in closed disks is not sufficient for the generalized Hadamard product set to be effective there. From this point of view we must restrict the coefficients in the original sets. We begin with Whittaker's condition [8].

Suppose that $\{p_n(z)\}$ and $\{q_n(z)\}$ are two simple monic sets according to Whittaker's condition in the form

$$|p_{n,j}| \leq M_1 \tag{3.2}$$

and

$$|q_{n,j}| \leq M_2;\ 0 \leq j \leq n-1. \tag{3.3}$$

Then

$$|p^{\mu}_{n,j}| \leq M^{\mu}_1 \tag{3.4}$$

and

$$|q^{\nu}_{n,j}| \leq M^{\nu}_2;\ 0 \leq j \leq n-1. \tag{3.5}$$

Their generalized Hadamard product set is given by

$$U^{(\mu,\nu)}_n(z) = \sum_j U^{(\mu,\nu)}_{n,j} z^j = \sum_{j=0}^{n-1} p^{\mu}_{n,j} q^{\nu}_{n,j} z^j + z^n \tag{3.6}$$

where

$$|U^{(\mu,\nu)}_{n,j}| = |p^{\mu}_{n,j} q^{\nu}_{n,j}| \leq M^{\mu}_1 M^{\nu}_2.$$

Since $\{U^{(\mu,\nu)}_n(z)\}$ is a simple monic set, then it satisfies the following relation [5]

$$U^{(\mu,\nu)}_{n+k,n+k} \bar{U}^{(\mu,\nu)}_{n+k,n} = -\sum_{j=0}^{k-1} U^{(\mu,\nu)}_{n+k,n+j} \bar{U}^{(\mu,\nu)}_{n+j,n}. \tag{3.7}$$

From which it follows, by mathematical induction that

$$|U^{(\mu,\nu)}_{n,j}| = M^{\mu}_1 M^{\nu}_2 (1 + M^{\mu}_1 M^{\nu}_2)^{n-j-1}. \tag{3.8}$$

Suppose that $r \geq (1 + M^{\mu}_1 M^{\nu}_2)$, then we have from (3.4), (3.5), and (3.6) that

$$M^{(\mu,\nu)}_n(r) = \max_{|z|=r} \left| \sum_k U^{(\mu,\nu)}_{n,k} z^k \right| \tag{3.9}$$

$$\leq \sum_{k=0}^{n-1} |p^{\mu}_{n,k}||q^{\nu}_{n,k}| r^k + r^n$$

$$\leq r^n + \sum_{k=0}^{n-1} M^{\mu}_1 M^{\nu}_2 r^k \leq 2r^n.$$

Substituting from (3.8) and (3.9) in the Cannon sum $\omega_n^{(\mu,\nu)}(r)$ of the generalized Hadamard product set, we get

$$\omega_n^{(\mu,\nu)}(r) = \sum_{k=0}^{n-1} |\bar{U}_{n,k}^{(\mu,\nu)}| M_k^{(\mu,\nu)}(r) + M_n^{(\mu,\nu)}(r) \tag{3.10}$$

$$\leq \sum_{k=0}^{n-1} M_1^{\mu} M_2^{\nu} (1 + M_1^{\mu} M_2^{\nu})^{n-k-1} 2r^k + 2r^n = K(n+1)r^n.$$

Then,

$$\omega^{(\mu,\nu)}(r) = \limsup_{n\to\infty} \{\omega_n^{(\mu,\nu)}(r)\}^{\frac{1}{n}} \leq r.$$

This means that the generalized Hadamard product set is effective in $|z| \leq r$ for all $r \geq (1 + M_1^{\mu} M_2^{\nu})$. Thus, we have the following.

Theorem 3.1. The generalized Hadamard product set is effective in every closed circle $|z| \leq r$ for all $r \geq (1 + M_1^{\mu} M_2^{\nu})$ whenever its original sets $\{p_n(z)\}$ and $\{q_n(z)\}$ are simple monic sets satisfying condition (3.2) and (3.3), respectively.

Putting $\mu = \nu = 1$ we have the following.

Corollary 3.1. The Hadamard product set of polynomials is effective in $|z| \leq r$ for all $r \geq 1 + M_1 M_2$ whenever the original sets $\{p_n(z)\}$ and $\{q_n(z)\}$ satisfy condition (3.2) and (3.3), respectively.

To show that the result in Theorem 3.1 is best possible in its domain, we give the following example.

Example 3.2. Let $\{p_n(z)\}$ and $\{q_n(z)\}$ be two simple sets given by

$$\begin{cases} p_0(z) = 1 \\ p_n(z) = z^n + M_1(z^{n-1} - z^{n-2} + z^{n-3} - \dots) \end{cases}$$

and

$$\begin{cases} q_0(z) = 1 \\ q_n(z) = z^n + M_2(z^{n-1} - z^{n-2} + z^{n-3} - \dots). \end{cases}$$

Take $\mu = 2$ and $\nu = 3$, then

$$\begin{cases} p_0^2(z) = 1 \\ p_n^2(z) = z^n + M_1^2(z^{n-1} - z^{n-2} + z^{n-3} - \dots) \end{cases}$$

and

$$\begin{cases} q_0^3(z) = 1 \\ q_n^3(z) = z^n + M_2^3(z^{n-1} - z^{n-2} + z^{n-3} - \dots). \end{cases}$$

It follows that their generalized Hadamard product set $\{U_n^{(2,3)}(z)\}$, whose original sets are $\{p_n(z)\}$ and $\{q_n(z)\}$ is given by

$$\begin{cases} U_0^{(2,3)}(z) = 1 \\ U_n^{(2,3)}(z) = z^n + M_1^2 M_2^3(z^{n-1} - z^{n-2} + z^{n-3} - \dots). \end{cases}$$

From this we see that

$$z^n = U_n^{(2,3)}(z) - M_1^2 M_2^3 \sum_{j=1}^{n} (-1)^{j-1}(1 + M_1^2 M_2^3)^{j-1} U_{n-j}^{(2,3)}(z).$$

Therefore, the Cannon sum $\omega_n^{(2,3)}(r)$ of this set is

$$\begin{aligned} \omega_n^{(2,3)}(r) &= r^n + 2M_1^2 M_2^3 r^{n-1} + 2M_1^2 M_2^3(1 + M_1^2 M_2^3) r^{n-2} \\ &\quad + 2M_1^2 M_2^3(1 + M_1^2 M_2^3)^2 r^{n-3} + 2M_1^2 M_2^3(1 + M_1^2 M_2^3)^3 r^{n-4} + \dots \\ &\quad + 2M_1^2 M_2^3(1 + M_1^2 M_2^3)^{n-1} \\ &= r^n + 2M_1^2 M_2^3 \sum_{j=1}^{n} (1 + M_1^2 M_2^3)^{j-1} r^{n-j}. \end{aligned}$$

Hence,

$$\omega^{(2,3)}(r) = \limsup_{n\to\infty} \{\omega_n^{(2,3)}(r)\}^{\frac{1}{n}} \geq M_1^2 M_2^3 + 1$$

whenever $r < 1+M_1^2M_2^3$ and the generalized Hadamard product set is not effective in $|z| \leq r$ for all $r < 1+M_1^2M_2^3$ as required.

Now we consider Boas's condition [1]. Suppose that the simple monic sets $\{p_n(z)\}$ and $\{q_n(z)\}$ satisfy Boas's condition in the form

$$|p_{n,j}| \leq M_1 a_1^{n-j} \tag{3.11}$$

and

$$|q_{n,j}| \leq M_2 a_2^{n-j};\ 0 \leq j \leq n-1. \tag{3.12}$$

Under these conditions, we have the following result.

Theorem 3.2. If the two simple monic sets $\{p_n(z)\}$ and $\{q_n(z)\}$ satisfy condition (3.11) and (3.12), respectively. Then the generalized Hadamard product set will be effective in $|z| \leq r$ for all $r \geq a_1^{\mu}a_2^{\nu}(1+M_1^{\mu}M_2^{\nu})$.

Proof. Suppose that μ and ν are any two positive numbers and the two simple monic sets $\{p_n(z)\}$ and $\{q_n(z)\}$ satisfy conditions (3.11) and (3.12), respectively, then

$$|p_{n,j}^{\mu}| \leq M_1^{\mu} a_1^{(n-j)\mu} \tag{3.13}$$

and

$$|q_{n,j}^{\mu}| \leq M_2^{\nu} a_2^{(n-j)\nu};\ 0 \leq j \leq n-1. \tag{3.14}$$

So the generalized Hadamard product set $\{U_n^{(\mu,\nu)}(z)\}$, where

$$U_n^{(\mu,\nu)}(z) = \sum_{k=0}^{n-1} U_{n,k}^{(\mu,\nu)} z^k + z^n = \sum_{k=0}^{n-1} p_{n,k}^{\mu} q_{n,k}^{\nu} z^k + z^n \tag{3.15}$$

satisfies the following condition

$$|U_{n,k}^{(\mu,\nu)}| = |p_{n,k}^{\mu}||q_{n,k}^{\nu}| \leq M_1^{\mu}M_2^{\nu}(a_1^{\mu}a_2^{\nu})^{n-k}. \tag{3.16}$$

Since the generalized Hadamard product set is a simple monic one, then it satisfies the following relation:

$$U_{n+k,n+k}^{(\mu,\nu)} \bar{U}_{n+k,n}^{(\mu,\nu)} = -\sum_{j=0}^{k-1} U_{n+k,n+j}^{(\mu,\nu)} \bar{U}_{n+j,n}^{(\mu,\nu)}.$$

Putting $k = 1$, $k = 2$, $k = 3$ and $k = 4$ successively, we have

$$|\bar{U}^{(\mu,\nu)}_{n+1,n}| = |U^{(\mu,\nu)}_{n+k,n}| \leq M_1^\mu M_2^\nu (a_1^\mu a_2^\nu),$$

$$\begin{aligned}|\bar{U}^{(\mu,\nu)}_{n+2,n}| &\leq |U^{(\mu,\nu)}_{n+2,n}| + |U^{(\mu,\nu)}_{n+2,n+1}||\bar{U}^{(\mu,\nu)}_{n+1,n}| \\ &\leq M_1^\mu M_2^\nu (a_1^\mu a_2^\nu)^2 + M_1^\mu M_2^\nu (a_1^\mu a_2^\nu) M_1^\mu M_2^\nu (a_1^\mu a_2^\nu) \\ &= M_1^\mu M_2^\nu (a_1^\mu a_2^\nu)^2 (1 + M_1^\mu M_2^\nu),\end{aligned}$$

$$\begin{aligned}|\bar{U}^{(\mu,\nu)}_{n+3,n}| &\leq |U^{(\mu,\nu)}_{n+3,n}| + |U^{(\mu,\nu)}_{n+3,n+1}||\bar{U}^{(\mu,\nu)}_{n+1,n}| + |U^{(\mu,\nu)}_{n+3,n+2}||\bar{U}^{(\mu,\nu)}_{n+2,n}| \\ &\leq M_1^\mu M_2^\nu (a_1^\mu a_2^\nu)^3 (1 + M_1^\mu M_2^\nu)^2\end{aligned}$$

and

$$\begin{aligned}|\bar{U}^{(\mu,\nu)}_{n+4,n}| &\leq |U^{(\mu,\nu)}_{n+4,n}| + |U^{(\mu,\nu)}_{n+4,n+1}||\bar{U}^{(\mu,\nu)}_{n+1,n}| + |U^{(\mu,\nu)}_{n+4,n+2}||\bar{U}^{(\mu,\nu)}_{n+2,n}| \\ &\quad + |U^{(\mu,\nu)}_{n+4,n+3}||\bar{U}^{(\mu,\nu)}_{n+3,n}| \\ &\leq M_1^\mu M_2^\nu (a_1^\mu a_2^\nu)^4 (1 + M_1^\mu M_2^\nu)^3.\end{aligned}$$

Proceeding in this manner, we arrive at the following relation for $k = 1, 2, \ldots, m$:

$$|\bar{U}^{(\mu,\nu)}_{n+m,k}| \leq M_1^\mu M_2^\nu (a_1^\mu a_2^\nu)^m (1 + M_1^\mu M_2^\nu)^{m-1}. \tag{3.17}$$

Then for $k = m + 1$, we have

$$\begin{aligned}|\bar{U}^{(\mu,\nu)}_{n+m+1,n}| &\leq |U^{(\mu,\nu)}_{n+m+1,n}| + \sum_{j=1}^{m} |U^{(\mu,\nu)}_{n+m+1,n+j}||\bar{U}^{(\mu,\nu)}_{n+j,n}| \\ &\leq M_1^\mu M_2^\nu (a_1^\mu a_2^\nu)^{m+1} (1 + M_1^\mu M_2^\nu)^m,\end{aligned}$$

i.e., relation (3.17) is true by mathematical induction.

Suppose that $r \geq a_1^\mu a_2^\nu(1 + M_1^\mu M_2^\nu)$. It follows that

$$M_n^{(\mu,\nu)}(r) = \max_{|z|=r} |U_n^{(\mu,\nu)}(z)| \tag{3.18}$$

$$\leq \sum_{j=0}^{n} U_{n,j}^{(\mu,\nu)} r^j$$

$$\leq r^n + \sum_{j=0}^{n-1} M_1^\mu M_2^\nu (a_1^\mu a_2^\nu)^{n-j} r^j$$

$$\leq r^n \left(1 + \frac{M_1^\mu M_2^\nu}{\frac{r}{a_1^\mu a_2^\nu} - 1}\right) \leq 2r^n.$$

Inserting (3.17) and (3.18) in the Cannon sum $\omega_n^{(\mu,\nu)}(r)$ of the generalized Hadamard product set we get

$$\omega_n^{(\mu,\nu)}(r) = \sum_{j=0}^{n} |\bar{U}_{n,j}^{(\mu,\nu)}| M_j^{(\mu,\nu)}(r) \tag{3.19}$$

$$= M_n^{(\mu,\nu)}(r) + \sum_{j=0}^{n-1} |\bar{U}_{n,j}^{(\mu,\nu)}| M_j^{(\mu,\nu)}(r)$$

$$= 2r^n + \sum_{j=0}^{n-1} M_1^\mu M_2^\nu (1 + M_1^\mu M_2^\nu)^{n-j-1} 2r^j$$

$$\leq k(n+1)r^n.$$

Then the Cannon function for the generalized Hadamard product set is

$$\omega^{(\mu,\nu)}(r) = \limsup_{n\to\infty} \{\omega_n^{(\mu,\nu)}(r)\}^{\frac{1}{n}} \leq r.$$

This completes the proof of the theorem.

Taking $\mu = \nu = 1$, we have the following corollary.

Corollary 3.2. If the two simple monic sets $\{p_n(z)\}$ and $\{q_n(z)\}$ satisfy condition (3.11) and (3.12), respectively, then the Hadamard product set of polynomials is effective in $|z| \leq r$, $r \geq a_1 a_2(1 + M_1 M_2)$.

The following example shows that the result in Theorem (3.2) is the best possible.

Example 3.3. Let $\{p_n(z)\}$ and $\{q_n(z)\}$ be two simple monic sets, where

$$\begin{cases} p_0(z) = 1 \\ p_n(z) = z^n + M_1(2z^{n-1} - 2^2 z^{n-2} + 2^3 z^{n-3} - \dots) \end{cases}$$

and

$$\begin{cases} q_0(z) = 1 \\ q_n(z) = z^n + M_2(3z^{n-1} - 3^2 z^{n-2} + 3^3 z^{n-3} - \dots). \end{cases}$$

Thus, the generalized Hadamard product set $\{U_n^{(4,5)}(z)\}$ will be such that

$$\begin{cases} U_0^{(4,5)}(z) = 1 \\ U_n^{(4,5)}(z) = z^n + M_1^4 M_2^5 (2^4 3^5 z^{n-1} - 2^8 3^{10} z^{n-2} + 2^{12} 3^{15} z^{n-3} - \dots). \end{cases}$$

The following representation is possible in the form

$$z^n = U_n^{(4,5)}(z) - M_1^4 M_2^5 \sum_{j=1}^{n} (-1)^{j-1} (2^4 3^5)^j (1 + M_1^4 M_2^5)^{j-1} U_{n-j}^{(4,5)}(z).$$

The Cannon sum of the generalized Hadamard product set is

$$\omega_n^{(4,5)}(r) = r^n + 2M_1^4 M_2^5 \sum_{j=1}^{n} (2^4 3^5)^j (1 + M_1^4 M_2^5)^{j-1} r^{n-j}.$$

Hence,

$$\omega_n^{(4,5)}(r) > 2M_1^4 M_2^5 (2^4 3^5)^n (1 + M_1^4 M_2^5)^{n-1}$$

and

$$\omega^{(4,5)}(r) = \limsup_{n\to\infty}\{\omega_n^{(4,5)}(r)\}^{\frac{1}{n}} \geq 2^4 3^5(1+M_1^4M_2^5) \text{ for all } r < 2^4 3^5(1+M_1^4M_2^5)$$

and the generalized Hadamard product set is effective in $|z| \leq r$ for all $r < 2^4 3^5(1+M_1^4M_2^5)$ as we want to prove.

Let the two simple monic sets $\{p_n(z)\}$ and $\{q_n(z)\}$ satisfy the following conditions [7].

$$|p_{n,j}| \leq \alpha_{n-j} \tag{3.20}$$

and

$$|q_{n,j}| \leq \beta_{n-j};\ 0 \leq j \leq n-1. \tag{3.21}$$

Suppose that μ and ν are any two positive numbers, then

$$|p^{\mu}_{n,j}| \leq \alpha^{\mu}_{n-j}, \tag{3.22}$$

$$|q^{\nu}_{n,j}| \leq \beta^{\nu}_{n-j};\ 0 \leq n-1 \tag{3.23}$$

and

$$H(R) = \sum_{j=1}^{\infty} \frac{\alpha_j^{\mu}\beta_j^{\nu}}{R^j} \leq 1. \tag{3.24}$$

If $\{U_n^{(\mu,\nu)}(z)\}$ is the generalized Hadamard product set of polynomials, then

$$|U_{n,j}^{(\mu,\nu)}| = |p^{\mu}_{n,j}||q^{\nu}_{n,j}| \leq \alpha^{\mu}_{n-j}\beta^{\nu}_{n-j} \tag{3.25}$$

$$= \frac{\alpha^{\mu}_{n-j}\beta^{\nu}_{n-j}}{R^{n-j}}R^{n-j} \leq R^{n-j}.$$

As above and by using relation (3.7) and mathematical induction we infer that

$$|\bar{U}_{n,j}^{\mu,\nu)}| \leq R^{n-j}. \tag{3.26}$$

From relation (3.25) we have

$$M_n^{(\mu,\nu)}(R) = \max_{|z|=R} |U_n^{(\mu,\nu)}(z)| \tag{3.27}$$

$$\leq \sum_{j=0}^{n} |U_{n,j}^{(\mu,\nu)}| R^j \leq R^n(1+n).$$

Inserting (3.26) and (3.27) in the Cannon sum $\omega_n^{(\mu,\nu)}(R)$ of the generalized Hadamard product set, we get

$$\omega_n^{(\mu,\nu)}(R) = \sum_{j=0}^{n} |\bar{U}_{n,j}^{(\mu,\nu)}| M_j^{(\mu,\nu)}(R) \tag{3.28}$$

$$\leq \sum_{j=0}^{n-1} |\bar{U}_{n,j}^{(\mu,\nu)}| M_j^{(\mu,\nu)}(R) + M_n^{(\mu,\nu)}(R)$$

$$\leq R^n(1+n)^2.$$

Taking the n–th root and making $n \to \infty$ we infer that $\omega_n^{(\mu,\nu)}(R) \leq R$ for all R and the generalized Hadamard product set is effective in $|z| \leq R$ for all R.

Therefore, the following theorem is completely established.

Theorem 3.3. Let $\{p_n(z)\}$ be a simple monic set of polynomials satisfying condition (3.20) and let the set $\{q_n(z)\}$ be a simple monic set of polynomials satisfying condition (3.21). Then the generalized Hadamard product set $\{U_n^{(\mu,\nu)}(z)\}$ of these two sets is effective in $|z| \leq R$ for all R.

Putting $\mu = \nu = 1$ we have the following corollary.

Corollary 3.3. Let $\{p_n(z)\}$ and $\{q_n(z)\}$ be two simple monic sets of polynomials satisfying condition (3.20) and (3.21), respectively. Then the Hadamard product set $\{U_n(z)\}$ of the two sets is effective in $|z| \leq R$.

The following example shows that this result is best possible for the effectiveness of a generalized Hadamard product set whose original sets satisfy conditions (3.20) and (3.21).

Example 3.4. Let $\{p_n(z)\}$ and $\{q_n(z)\}$ be two simple monic sets of polynomials given by

$$p_0(z) = 1,\ p_n(z) = z^n + \frac{1}{2\sqrt{n}};\ (n > 0)$$

and

$$q_0(z) = 1,\ q_n(z) = z^n + \frac{1}{3\sqrt{n}};\ (n > 0).$$

Then $p_{n,j} = 0$ for all $j \neq 0,\ p_{n,0} = \frac{1}{2\sqrt{n}} = \alpha_n$ and $q_{n,j} = 0$ for all $j \neq 0,\ q_{n,0} = \frac{1}{3\sqrt{n}} = \beta_n$.

Now, we construct the generalized Hadamard product set $\{U_n^{(2,2)}(z)\}$ as follows:

$$U_0^{(2,2)}(z) = 1,\ U_n^{(2,2)}(z) = z^n + \frac{1}{36n^2},$$

so that z^n has the following representation by generalized Hadamard product set in the form

$$z^n = U_n^{(2,2)}(z) - \frac{1}{36n^2} U_0^{(2,2)}(z).$$

From which the Cannon sum $\omega_n^{(2,2)}(R)$ of the generalized Hadamard product set is given by

$$\omega_n^{(2,2)}(R) = R^n + \frac{1}{18n^2}. \tag{3.29}$$

The Cannon function $\omega^{(2,2)}(R)$ is such that

$$\omega^{(2,2)}(R) = \begin{cases} 1 & R < 1 \\ R & R \geq 1, \end{cases}$$

i.e., the set $\{U_n^{(2,2)}(z)\}$ is not effective in $|z| \leq R$ for all $R < 1$, but it is effective in $|z| \leq R$ for all $R \geq 1$ as required.

Now, we show that Theorem 3.1 and 3.2 follow from Theorem 3.3.

At first, let R be any positive number $\geq 1 + M_1^\mu M_2^\nu$. Then for $\alpha_j \leq M_1$ and $\beta_j \leq M_2$ we have $\sum_{j=1}^{\infty} \frac{\alpha_j^\mu \beta_j^\nu}{R^j} \leq M_1^\mu M_2^\nu \sum_{j=1}^{\infty} (\frac{1}{R})^j = M_1^\mu M_2^\nu \frac{1}{R-1} \leq 1$. Therefore, (3.24) follows and the generalized Hadamard product set is effective in $|z| \leq R$ for all $R \geq 1 + M_1^\mu M_2^\nu$.

Also, if we take $R \geq a_1^\mu a_2^\nu (1 + M_1^\mu M_2^\nu)$, $\alpha_j \leq M_1 a_1^j$, $\beta_j \leq M_2 a_2^j$ and do the same steps as above we see that the generalized Hadamard product set is effective in $|z| \leq R$ for all $R \geq a_1^\mu a_2^\nu (1 + M_1^\mu M_2^\nu)$ and Theorem (3.2) follows.

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K. A. M. Sayyed
Department of Mathematics
Faculty of Science
University of Assiut
Assiut 71516, EGYPT

M. S. Metwally
Department of Mathematics
Faculty of Science
University of Assiut
Assiut 71516, EGYPT
email: met641958@yahoo.com

M. T. Mohamed
Department of Mathematics
Faculty of Science
University of Assiut
Assiut 71516, EGYPT