SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

143. [2003, 201; 2004, 129] *Proposed by José Luis Diaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let α, β, and γ be the angles of acute triangle ABC. Prove that

$$\frac{\cot\alpha\cot\beta}{\sqrt{1-\cot\alpha\cot\beta}}+\frac{\cot\beta\cot\gamma}{\sqrt{1-\cot\beta\cot\gamma}}+\frac{\cot\gamma\cot\alpha}{\sqrt{1-\cot\gamma\cot\alpha}}\geq\sqrt{\frac{3}{2}}.$$

Solution by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan. First we notice that the expressions under the square roots are positive since

$$1-\cot\alpha\cot\beta=\frac{-\cos(\alpha+\beta)}{\sin\alpha\sin\beta}\geq 0,$$

since $\sin\alpha,\sin\beta>0$ and $\cos(\alpha+\beta)\leq 0$. (i.e. $\alpha+\beta\geq 90°$; $180°-\gamma\geq 90°$ implies $\gamma\leq 90°$). I'll make use of the following equality which holds in any triangle:

$$\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1,$$

where α, β, and γ are the angles of a given triangle ABC. Denote

$$a=\cot\alpha\cot\beta,\quad b=\cot\beta\cot\gamma,\quad c=\cot\gamma\cot\alpha.$$

Let

$$f\colon(0,1)\to\mathbb{R},\quad\text{where}\quad f(x)=\frac{x}{\sqrt{1-x}}.$$

Then

$$f'(x)=\frac{2-x}{2(1-x)^{\frac{3}{2}}}\quad\text{and}\quad f''(x)=\frac{4-x}{4(1-x)^{\frac{5}{2}}}>0.$$

Therefore, f is a convex function on $(0,1)$. By applying Jensen's Inequality to f we get that

$$f\left(\frac{a+b+c}{3}\right) \le \frac{f(a)+f(b)+f(c)}{3}.$$

Therefore,

$$\frac{a}{\sqrt{1-a}} + \frac{b}{\sqrt{1-b}} + \frac{c}{\sqrt{1-c}} \ge \frac{3 \cdot \frac{a+b+c}{3}}{\sqrt{1-\frac{a+b+c}{3}}} = \frac{1}{\sqrt{1-\frac{1}{3}}} = \sqrt{\frac{3}{2}};$$

since $a+b+c=1$.

Also solved by Joe Howard, Portales, New Mexico; Mihai Cipu, Romanian Academy, Bucharest, Romania; and the proposer.