JAPANESE THEOREM: A LITTLE KNOWN THEOREM WITH MANY PROOFS. (PART II)

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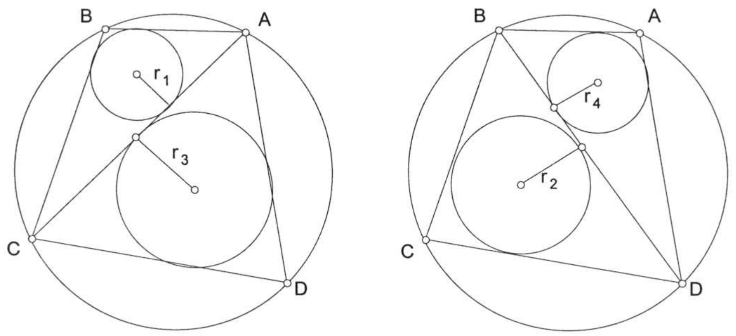
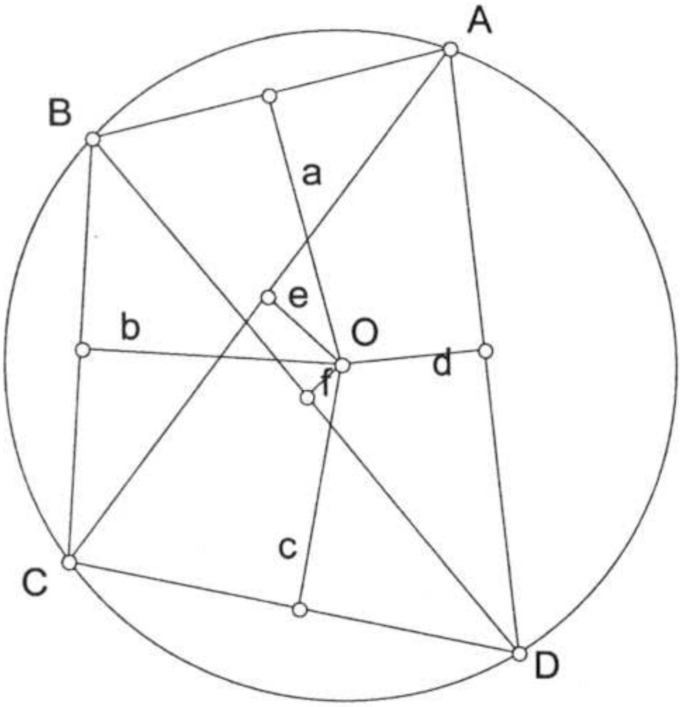
1. Later Attempts. By early 20th century Japanese mathematics had flourished and papers by Japanese mathematicians began to appear in western journals. In a 1906 paper in *Mathesis* [4], Prof. T. Hayashi conveyed no less than five different proofs of our theorem by Japanese mathematicians. To exhibit the rich variety of approaches to the theorem, all five proofs are presented here. While some proofs are easy, and others require a little patience, they all testify to the level of sophistication of Japanese mathematics at that time. Readers should refer to Part I for results (E1) to (E5) and (G1) to (G8).

Japanese Theorem (Quadrilateral Case). Let $ABCD$ be a quadrilateral inscribed in a circle. Let r_1, r_2, r_3, and r_4 be the radii of the circles C_1, C_2, C_3, and C_4 inscribed in the triangles ABC, BCD, CDA, and DAB, respectively. Then $r_1 + r_3 = r_2 + r_4$.

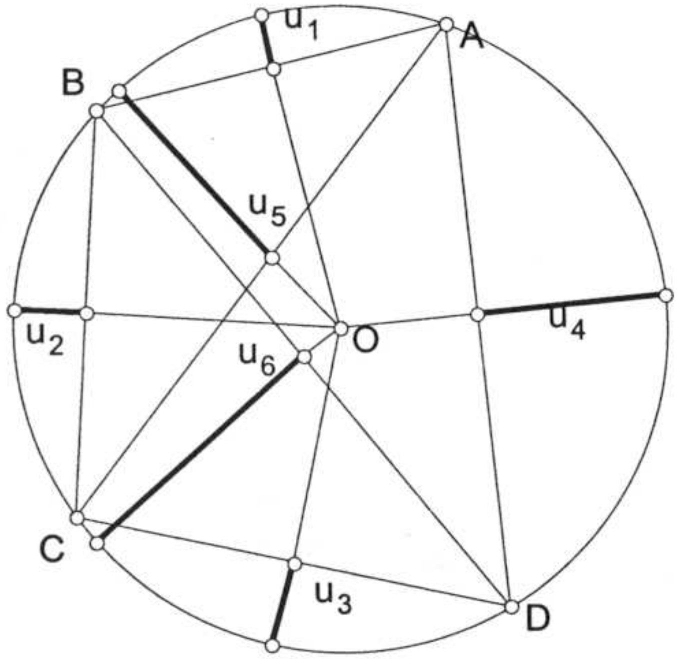


Nagasawa's Proof. (Kamenosuke Nagasawa was born in 1860 and graduated from college in 1878. He had written 150 books and translations before his death in 1927.)



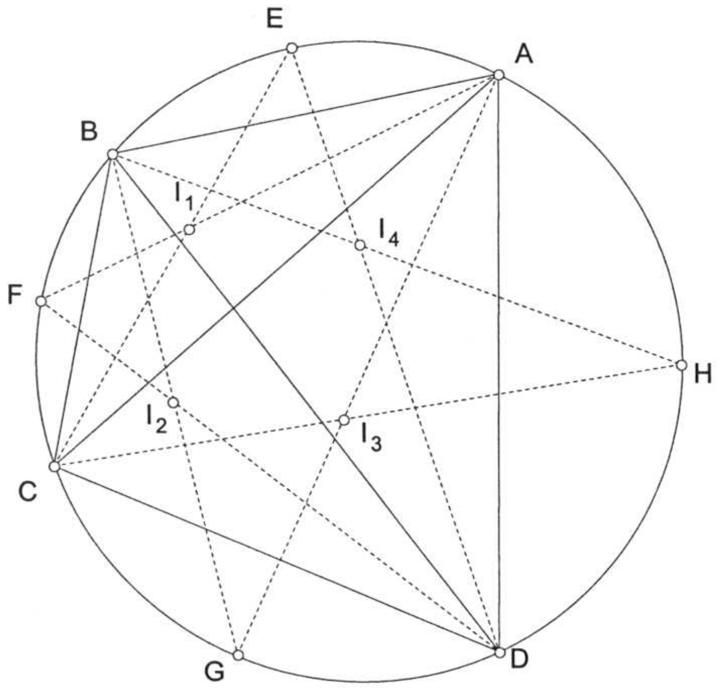
Let a, b, c, d, e, and f denote the lengths of the perpendiculars from O to lines AB, BC, CD, DA, AC, and BD, respectively. Using (G4) on triangles ABC and CDA, we get $R + r_1 = a + b - e$, and $R + r_3 = c + d + e$. On adding, we have $2R + r_1 + r_3 = a + b + c + d$. Similarly, from triangles BCD and DAB we get $2R + r_2 + r_4 = a + b + c + d$. On equating the two results, we get $r_1 + r_3 = r_2 + r_4$.

Proof by Sawayama. (Yuzaburo Sawayama (1860–1936) was a professor in the Japanese army and later at the Tokyo Physics College.)



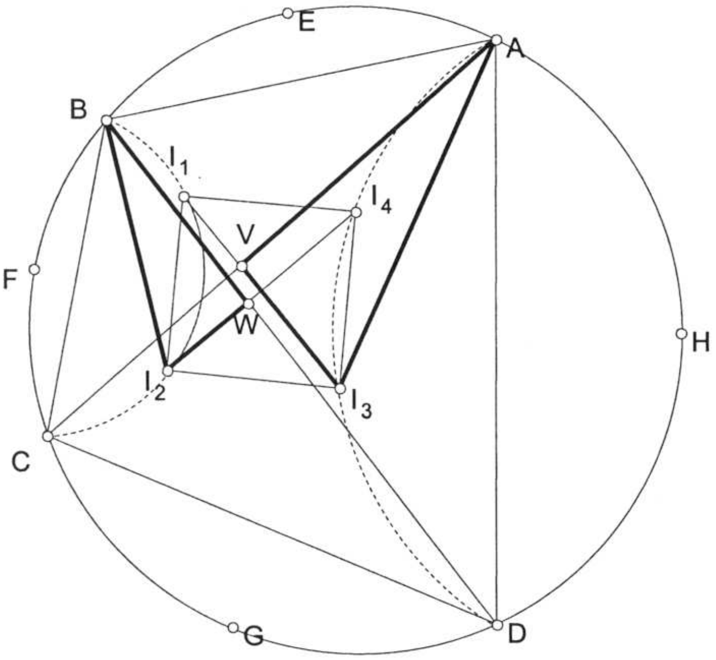
Let a, b, c, d, e, and f denote perpendicular from O to lines AB, BC, CD, DA, AC, and BD, respectively. Let us extend the perpendicular, say from O to AB, to reach the midpoint of the arc AB. Let u_1 denote the length from the midpoint of the chord AB to the midpoint of the arc AB. Let u_2, u_3, u_4, u_5, and u_6 be similarly defined for the chords BC, CD, DA, AC, and BD, respectively. Note that $u_1 = R - a$, $u_2 = R - b$, and so on. From triangle ABC, $u_1 + u_2 + (2R - u_5) = (R - a) + (R - b) + (R + e) = 3R - (a + b - e)$, which by (G4) equals $3R - (R + r_1) = 2R - r_1$. Similarly, from triangle CDA we get $u_3 + u_4 + u_5 = (R-c)+(R-d)+(R-e) = 3R-(c+d+e) = 3R-(R+r_3) = 2R-r_3$. On adding these two, we get $u_1+u_2+u_3+u_4+2R = 4R-(r_1+r_3)$, or $u_1+u_2+u_3+u_4 = 2R-(r_1+r_3)$. Similarly, from triangles BCD and DAB we get $u_1+u_2+u_3+u_4 = 2R-(r_2+r_4)$. On equating the two we get $r_1 + r_3 = r_2 + r_4$.

Proof by Nozaki. (Very little is known about Tsunezo Nozaki. His proof below uses result (G7) which says that $I_1I_2I_3I_4$ is a rectangle.)



If I_1, I_2, I_3, and I_4 denote the incenters of triangles ABC, BCD, CDA, and DAB, respectively, then by (G7) we know that the figure $I_1I_2I_3I_4$ is a rectangle. Using (G6) we get $OI_1^2 + OI_3^2 = OI_2^2 + OI_4^2$. But for each i, $OI_i^2 = R^2 - 2Rr_i$ by (G3). This means $R^2 - 2Rr_1 + R^2 - 2Rr_3 = R^2 - 2Rr_2 + R^2 - 2Rr_4$, which simplifies to $r_1 + r_3 = r_2 + r_4$.

Proof by Matsuo and Omori. (Very little is known about the two authors. Their proof hinges on a clever observation that the projection of I_1I_3 perpendicular to AC is exactly $r_1 + r_3$. Since $I_1I_3 = I_2I_4$ we only need to show that I_1I_3 and AC intersect at the same angle as I_2I_4 and BD.)



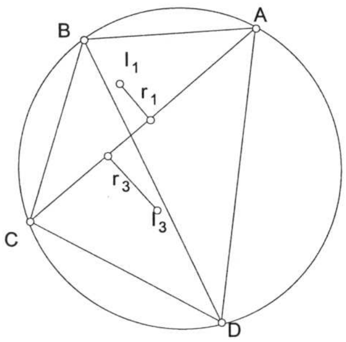
Let AC intersect I_1I_3 in V and BD intersect I_2I_4 in W. Angles AVI_3 and BWI_2, will be equal if we can show that the other two angles of triangles AVI_3 and BWI_2 are equal.

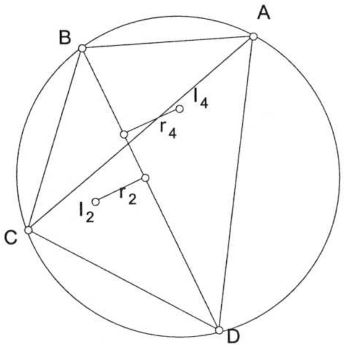
First, $\angle VAI_3 = \frac{1}{2}(\angle CAD) = \frac{1}{2}(\angle CBD) = \angle I_2BW$.

Secondly, to prove that the angles VI_3A and WI_2B are equal, we note that $\angle VI_3A = \angle VI_3I_4 + \angle I_4I_3A$, and $\angle WI_2B = \angle WI_2I_1 + \angle I_1I_2B$. But, in the rectangle $I_1I_2I_3I_4$ the angles VI_3I_4 and WI_2I_1 are equal. So we only need to show that $\angle I_4I_3A = \angle I_1I_2B$.

From (G2) we know that the points A, I_4, I_3, and D lie on a circle with center H. Also, the points B, I_1, I_2, and C lie on a circle with center F. So, $\angle I_4I_3A = \angle I_4DA = \frac{1}{2}(\angle BDA) = \frac{1}{2}(\angle BCA) = \angle BCI_1 = \angle I_1I_2B$.

Thus, triangles AVI_3 and BWI_2 are similar and $\angle AVI_3 = \angle BWI_2$.



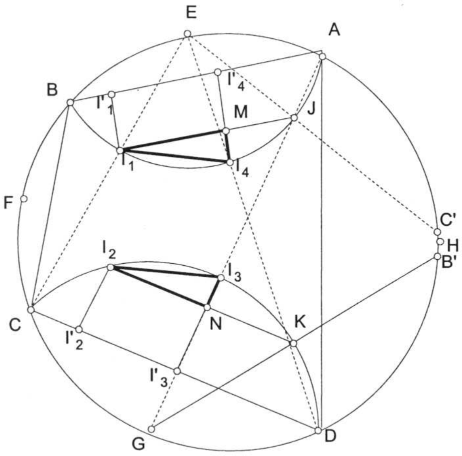


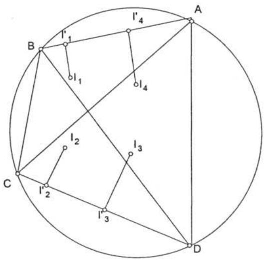
Since $I_1I_3 = I_2I_4$, and I_1I_3 cuts AC at the same angle as I_2I_4 cuts BD, the projection of I_1I_3 perpendicular to AC equals the projection of I_2I_4 perpendicular to BD. These projections being exactly r_1+r_3 and r_2+r_4, we have $r_1+r_3 = r_2+r_4$.

Proof by Chou. (Chou's full name is Shu Tatsu. Chou does not use the fact that $I_1I_2I_3I_4$ is a rectangle. Instead, his proof shows that $r_3 - r_2$, the difference of perpendiculars from I_2 and I_3 on CD, equals $r_4 - r_1$.)

Let I_1, I_2, I_3, and I_4 be the incenters of the triangles ABC, BCD, CDA, and DAB, respectively. We drop perpendiculars from I_1 and I_4 to side AB, meeting AB at I_1' and I_4'. Similarly, let I_2I_2' and I_3I_3' be perpendiculars to CD. We will note that $I_1I_1' = r_1$, $I_4I_4' = r_4$, $I_2I_2' = r_2$, and $I_3I_3' = r_3$. We want to show that $r_4 - r_1 = r_3 - r_2$. Let E, F, G, and H be the midpoints of the arcs AB, BC, CD, and DA, respectively. Also, let B' and C' be points on the circle such that arc GB = arc GB', and arc EC = arc EC'. Since arcs EA and EB are equal and arcs EC and EC' are equal, we have arc AC' = arc BC. Similarly, we find arc DB' = arc BC. Hence, the arcs AC' and DB' are equal, and by adding a piece of arc $B'C'$ to both, we have arc AB' = arc DC' and hence, $\angle AGB' = \angle DEC'$.

Let E, F, G, H be the midpoints of the arcs AB, BC, CD, DA, respectively. From (G2) we know that a circle with center E passes through the points A, I_4, I_1, and B, and a circle with center G passes through the points C, I_2, I_3, and D. Let C_E and C_G denote these two circles. Let circle C_E intersect EC' at J, and circle C_G intersect GB' at K. Let I_1J intersect I_4I_4' at M and I_2K intersect I_3I_3' at N. The idea behind these constructions are to show that $r_4 - r_1 = I_4M = I_3N = r_3 - r_2$.





We will prove that the triangles I_1MI_4 and I_2NI_3 (i) are right angled at M and N, (ii) are similar because $\angle MI_1I_4 = \angle I_3I_2N$, and (iii) are congruent because $I_1M = I_2N$.

To prove (i), we see that arc $EA =$ arc EB, and arc $EC =$ arc EC'. Hence, AB is parallel to CC'. Also, from triangle ECC', we have $EC = EC'$, and $EI_1 = EJ$, hence, I_1J is parallel to CC' and also to AB. Thus, $I_4I'_4$ is perpendicular to I_1J, and triangle I_1MI_4 is right angled at M. Similarly, triangle I_2NI_3 is right angled at N.

To prove (ii), we see that $\angle MI_1I_4 = \angle JI_1I_4 = \frac{1}{2}(\angle JEI_4) = \frac{1}{2}(\angle AGB') = \frac{1}{2}(\angle I_3GK) = \angle I_3I_2N$.

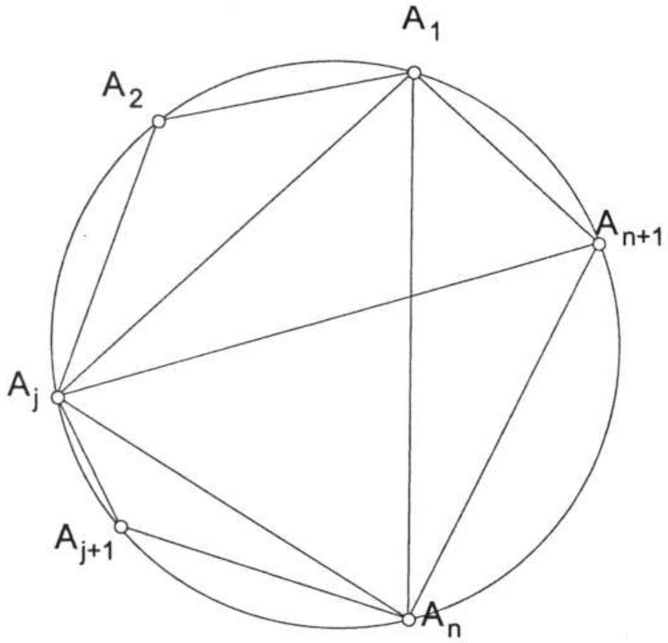
To prove (iii), we note that $I_1M = I'_1I'_4$, and $I_2N = I'_2I'_3$. Now $I'_1I'_4 = AI'_1 - AI'_4$, and by (G8) equals $\frac{1}{2}(AB + AC - BC) - \frac{1}{2}(AB + AD - BD) = \frac{1}{2}(AC + BD - BC - AD)$. Doing similar work, $I'_2I'_3 = CI'_3 - CI'_2 = \frac{1}{2}(CD + AC - AD) - \frac{1}{2}(CD + BC - BD) = \frac{1}{2}(AC + BD - BC - AD)$. Hence, $I_1M = I_2N$.

This shows that triangles I_1MI_4 and I_2NI_3 are congruent and $I_4M = r_4 - r_1$ is equal to $I_3N = r_3 - r_2$. Hence, $r_1 + r_3 = r_2 + r_4$.

2. Generalization. While the quadrilateral case came to Japan from China, it was Y. Mikami from Japan who generalized the theorem from the quadrilateral to a polygon [6]. Instead of showing how some of the proofs already shown here would work in the case of a polygon, we will show two different proofs using induction.

One easy way is to choose two non–adjacent vertices A_j and A_k, draw diagonal A_jA_k, which will divide the polygon into two smaller polygons, and then use the induction hypothesis on the two smaller polygons.

Here is another way of showing that the theorem holds even when we add an extra vertex to the polygon.



Let P be a polygon with n vertices $A_1, \ldots, A_n$, and let Q be a polygon with vertices $A_1, \ldots, A_n, A_{n+1}$. Let $S_j(P)$ denote the sum of the radii of the incircles of P when triangulation is done from vertex A_j. Also, let $r[ABC]$ denote the inradius of triangle ABC. As for S_1, obviously $S_1(Q) = S_1(P) + r[A_nA_{n+1}A_1]$. What about S_j?

We see that $S_j(Q) = S_j(P) - r[A_jA_nA_1] + r[A_jA_nA_{n+1}] + r[A_jA_{n+1}A_1]$. But from the quadrilateral case we have $r[A_jA_nA_1] + r[A_nA_{n+1}A_1] = r[A_jA_nA_{n+1}] + r[A_jA_{n+1}A_1]$. Hence, $S_j(Q) = S_j(P) + r[A_nA_{n+1}A_1]$.

Now to prove the general case by induction, let A_j and A_k be any two vertices. By the induction hypothesis, $S_j(P) = S_k(P)$. Then, by adding $r[A_nA_{n+1}A_1]$ to each we get $S_j(Q) = S_k(Q)$. This provides us with the inductive jump to go from P to Q.

3. Conclusion. One wonders why the Japanese theorem, which is not so well known even in Japan [8], has so many proofs. While only a few proofs look similar, others take us through different paths leading to the final summit. But they all display the power of classical plane geometry, used this time by the people from the Far East. On reading the different proofs one cannot but develop respect and admiration for the Japanese mathematicians of that time.

We hope that this paper inspires others to explore and unfold the story behind other forgotten theorems. Finally, we hope that it encourages fellow mathematicians to collaborate in similar multicultural projects.

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