PSEUDORESOLVENTS IN BANACH ALGEBRAS

Árpád Bényi and Bryan Dawson

Abstract. We give a sufficient condition for a family of pseudoresolvents in a Banach algebra to be trivially zero. As an important consequence, we provide an alternate proof of the classical result that the spectrum of any linear bounded operator on a Banach space is nonempty. The proofs are elementary, requiring only a basic knowledge of real and complex analysis.

1. Notation and Preliminaries. Let $\mathbf{A}$ denote a complex Banach Algebra, i.e., $\mathbf{A}$ is a linear space over the field of complex numbers $\mathbf{C}$ endowed with a complete norm $\|\cdot\|$ and a product $\mathbf{A}\times\mathbf{A} \ni (x,y) \mapsto xy \in \mathbf{A}$ such that the following properties hold:

(1) $(xy)z = x(yz)$ (associativity),

(2) $x(y+z) = xy + xz$ and $(y+z)x = yz + zx$ (distributivity),

(3) $(\alpha x)(\beta y) = (\alpha\beta)(xy)$,

(4) $\|xy\| \leq \|x\| \, \|y\|$,

(5) There exists an $e \in \mathbf{A}$ such that $xe = ex = x$ and $\|e\| = 1$,

for all $x, y \in \mathbf{A}$ and $\alpha, \beta \in \mathbf{C}$. The condition (5) on the unit is sometimes omitted from the definition of a Banach algebra. However, there is no loss of generality in this omission, since the Banach algebras without a unit can be endowed with one in a standard way; for more details, see [3].

We say that two pairs $(\alpha, x), (\beta, y) \in \mathbf{C}\times\mathbf{A}$ are equivalent, and write $(\alpha, x) \sim (\beta, y)$, if the following equalities hold:

$$(\alpha, x) \sim (\beta, y) \iff x - y = (\beta - \alpha)xy = (\beta - \alpha)yx.$$

For example, if T is a bounded linear operator on some Banach space, we have

$$(\lambda_1, (\lambda_1 I - T)^{-1}) \sim (\lambda_2, (\lambda_2 I - T)^{-1}),$$

and if $(\lambda_1, S) \sim (\lambda_2, (\lambda_2 I - T)^{-1})$ then S is the inverse of $\lambda_1 I - T$. Here, I denotes the identity operator.

Let us denote by $\tilde{\Re}$ the set of all the equivalence classes induced by $\sim$. For $Z \in \tilde{\Re}$, we define its *resolvent set*

$$\rho(Z) = \{\lambda \in \mathbf{C} \mid \text{ there exists a } z \in \mathbf{A} \colon (\lambda, z) \in Z\}.$$

Also, if $(\lambda, z) \in Z$, we write $z = R(\lambda, Z)$. The *spectrum set* is the complement in $\mathbf{C}$ of the resolvent set [1].

Now let $\Lambda \subseteq \mathbf{C}$. We say that $\{J(\lambda)\}_{\lambda\in\Lambda} \subseteq \mathbf{A}$ is a *family of pseudoresolvents over* Λ [2], if for all $\lambda_1, \lambda_2 \in \Lambda$ we have

$$J(\lambda_1) - J(\lambda_2) = (\lambda_2 - \lambda_1)J(\lambda_1)J(\lambda_2) = (\lambda_2 - \lambda_1)J(\lambda_2)J(\lambda_1).$$

For example, if $\mathbf{A} = \mathbf{C}$, $\Lambda = \mathbf{C}\backslash\{0\}$ and $J(\lambda) = \lambda^{-1}$, then $\{J(\lambda)\}_{\lambda\in\Lambda}$ is a family of pseudoresolvents. Note also that if $\{J(\lambda)\}_{\lambda\in\Lambda}$ is a family of pseudoresolvents then $(\lambda_1, J(\lambda_1)) \sim (\lambda_2, J(\lambda_2))$, for all $\lambda_1, \lambda_2 \in \Lambda$. Hence, all the pairs $(\lambda, J(\lambda)) \in \Lambda\times\mathbf{A}$ lie in the same equivalence class $Z_0 \in \tilde{\Re}$, $\rho(Z_0) \supseteq \Lambda$ and $J(\lambda) = R(\lambda, Z_0)$, for all $\lambda \in \Lambda$.

In what follows, we provide an answer to the following question:

In which conditions is a family of pseudoresolvents over $\mathbf{C}$ *trivially zero?*

As a consequence we will recover a classical result in functional analysis about the spectrum of a linear bounded operator on a Banach space.

2. The Main Result — Application. We first prove the following

Lemma. Let $\{J(\lambda)\}_{\lambda\in\Lambda}$ be a family of pseudoresolvents over the open and unbounded set Λ such that

(C1) $\|J(\lambda_1)\|\ \|J(\lambda_2)\| \leq M\|J(\lambda_1)J(\lambda_2)\|$, for all $\lambda_1, \lambda_2 \in \mathbf{C}$ (M is some positive constant); then there exists an open and bounded set $\Lambda_0 \subseteq \Lambda$ such that $\{\|J(\lambda)\|\}_{\lambda\in\Lambda\backslash\Lambda_0}$ is a bounded subset of $[0, \infty)$.

Proof. Let $\lambda_1, \lambda_2 \in \Lambda$. We have

$$\|J(\lambda_1)\| + \|J(\lambda_2)\| \geq \|J(\lambda_1) - J(\lambda_2)\| = |\lambda_1 - \lambda_2|\ \|J(\lambda_1)J(\lambda_2)\|$$

$$\geq \frac{|\lambda_1 - \lambda_2|}{M}\|J(\lambda_1)\|\ \|J(\lambda_2)\|.$$

Let $l = \limsup_{|\lambda|\to\infty}\|J(\lambda)\| \in [0, \infty]$. We wish to show that $l < \infty$. Assume by way of contradiction that this is not the case, i.e., $l = \infty$. Fix $\lambda_2 \in \Lambda$ such that $\|J(\lambda_2)\| > 2$ (if no such λ_2 exists, then $\|J(\lambda)\| \leq 2$ for all $\lambda \in \Lambda$ and the conslusion of the lemma trivially holds, with $\Lambda_0 = \emptyset$). Our assumption on l implies that there exists $\lambda_1 \in \Lambda$ such that both $|\lambda_1 - \lambda_2| > M$ and $\|J(\lambda_1)\| > \|J(\lambda_2)\|$. Consider now $k = \|J(\lambda_1)\|/\|J(\lambda_2)\| > 1$. Using now the above inequality, we obtain

$$(k+1)\|J(\lambda_2)\| = \|J(\lambda_1)\| + \|J(\lambda_2)\| \geq \frac{|\lambda_1 - \lambda_2|}{M}\|J(\lambda_1)\| \; \|J(\lambda_2)\|$$

$$> k\|J(\lambda_1)\| \; \|J(\lambda_2)\| > 2k\|J(\lambda_2)\|.$$

Hence, $k+1 > 2k$, or $k < 1$, a contradiction. Thus, $l < \infty$, which implies the desired conclusion.

The main result is given by the following proposition.

Proposition 1. If the family of pseudoresolvents $\{J(\lambda)\}_{\lambda \in \mathbf{C}}$ satisfies the conditions

(C1) $\|J(\lambda_1)\| \; \|J(\lambda_2)\| \leq M\|J(\lambda_1)J(\lambda_2)\|$, for all $\lambda_1, \lambda_2 \in \mathbf{C}$ (M is some positive constant);

(C2) $\mathbf{C} \ni \lambda \mapsto J(\lambda) \in \mathbf{A}$ is continuous, then $\{J(\lambda)\}_{\lambda \in \mathbf{C}} = \{0\}$.

Proof. The definition of the pseudoresolvent and (C2) show that

$$\mathbf{C} \ni \lambda \mapsto J(\lambda) \in \mathbf{A}$$

is holomorphic and $J'(\lambda) = -J^2(\lambda)$, for all $\lambda \in \mathbf{C}$. The previous lemma shows that $\{\|J(\lambda)\|\}_{\lambda \in \mathbf{C}}$ is bounded on the exterior of an open ball. The same condition (C2) shows that this set is bounded on this ball. Thus, the holomorphic map $\mathbf{C} \ni \lambda \mapsto J(\lambda) \in \mathbf{A}$ is also bounded on $\mathbf{C}$. Using Liouville's theorem [3] we conclude that this map must be constant, and since its limit at infinity is zero, this constant must be zero itself. This completes the proof of our proposition.

Let now X be a Banach space and $\mathbf{A} = \mathcal{B}(\mathcal{X})$ be the Banach algebra of linear bounded operators on X (with the operatorial norm). Note that the resolvent family of any linear bounded operator is a particular case of a family of pseudoresolvents.

Proposition 2. The spectrum of any linear bounded operator $T \in \mathbf{A}$ is nonempty.

Proof. By way of contradiction, assume that the spectrum is empty, i.e., $\rho(T) = \mathbf{C}$. Let us show that the resolvent family of the operator T satisfies the condition

(C1) of Proposition 1. More exactly, we will prove that for any $M > 1$ the following inequality holds:

$$(6) \quad \|R(\lambda_1,T)\| \, \|R(\lambda_2,T)\| \leq M\|R(\lambda_1,T)R(\lambda_2,T)\|,$$

where $\{R(\lambda,T) = (\lambda I - T)^{-1}\}_{\lambda\in\mathbf{C}}$ is the resolvent family of the operator T and

$$\lambda_1, \lambda_2 \in \Lambda = \left\{ z \in \mathbf{C} \mid |z| > \frac{\sqrt{M}+1}{\sqrt{M}-1}\|T\| \right\}.$$

Indeed, since

$$\|(\lambda I - T)\| \, \|R(\lambda,T)\| \leq \frac{|\lambda| + \|T\|}{|\lambda| - \|T\|} < \sqrt{M}, \ \lambda \in \Lambda,$$

we have

$$\|(\lambda_1 I - T)(\lambda_2 I - T)\| \, \|R(\lambda_1,T)\| \, \|R(\lambda_2,T)\| \leq (\sqrt{M})^2 = M, \ \lambda_1, \lambda_2 \in \Lambda.$$

Hence,

$$\|R(\lambda_1,T)R(\lambda_2,T)\| \geq \frac{1}{\|(\lambda_1 I - T)(\lambda_2 I - T)\|} \geq \frac{\|R(\lambda_1,T)\| \, \|R(\lambda_2,T)\|}{M},$$

which proves (6). It is known, however, that the map

$$\mathbf{C} \ni \lambda \mapsto R(\lambda,T) \in \mathbf{A}$$

is continuous. Using the above lemma, we conclude that the resolvent family $\{R(\lambda,T)\}_{\lambda\in\mathbf{C}}$ is bounded on $\mathbf{C}\backslash\Lambda_0$, where Λ_0 is an open and bounded subset of $\mathbf{C}$. Thus, the resolvent family is bounded on the exterior of the ball $B(0,R) \supseteq B(0, \frac{\sqrt{M}+1}{\sqrt{M}-1}) \cup \Lambda_0$. From this point on, an argument similar to the one in Proposition 1 shows that $\{R(\lambda,T)\}_{\lambda\in\mathbf{C}} = \{0\}$, a condradiction. The proof is complete.

Corollary. Under the hypotheses of Proposition 1, the family of pseudoresolvents $\{J(\lambda)\}_{\lambda\in\mathbf{C}}$ cannot be the resolvent family of any linear and bounded operator.

Remark. It can be shown that the spectrum of any linear bounded operator is a compact set as well. For more details see [3].

References :

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Árpád Bényi
Department of Mathematics
University of Kansas
Lawrence, KS 66045
email: benyi@math.ukans.edu

Bryan Dawson
Department of Mathematics
Union University
Jackson, TN 38305
email: ldawson@uu.edu