A NEW GENERALIZATION OF SHANIN'S NOTION of $\mathbf{R_0}$ TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and investigate some weak separation axioms by using the notions of (Λ, α)–open sets and the (Λ, α)–closure operator.

1. Introduction. In 1943, N. A. Shanin [8] offered a new weak separation axiom called R_0 to the world of general topology. In 1961, A. S. Davis [2] rediscovered this axiom and established some properties of topological spaces endowed with it. Later on several topologists further investigated R_0 topological spaces [3,4,5,6]. The notion of α–open set was introduced by O. Njåstad [7] in 1965. Since then it has been investigated in different respects in the literature. Quite recently Caldas, et. al. [1] introduced and studied the notions of (Λ, α)–open sets, (Λ, α)–closed sets, and the (Λ, α)–closure operator. This paper deals with some new low separation axioms by utilizing (Λ, α)–open and (Λ, α)–closed sets. There is no doubt that low separation axioms play a very important role in general topology. Indeed, there are lots of research papers which deal with different low separation axioms and also many topologists worldwide are doing research in this area. It is the aim of this paper to offer some new types of low separation axioms by using (Λ, α)–open sets and (Λ, α)–closure operator.

In this paper, by (X, τ) and (Y, σ) (or X and Y) we always mean topological spaces. Let A be a subset of X. The subset A of the topological space (X, τ) is called *α–open* (originally called *α-sets*) [7] if $A \subseteq Int(Cl(Int(A)))$. The complement of a α–open set is called α–closed. By $\alpha\mathrm{O}(X, \tau)$ (respectively $\alpha\mathrm{C}(X, \tau)$), we denote the family of all α–open (respectively α–closed) sets of (X, τ). Observe that α–open sets form a topology, and also α–openness does not imply openness in the underlying topology. Let C denote the standard "middle thirds" Cantor set in the unit interval $[0, 1]$ with the standard topology, and let $x \in C$. Then, $[0, 1] - \{C - \{x\}\}$ would be an α–open set. The intersection of all α–closed sets containing A is called the α–closure of A, denoted by $Cl_\alpha(A)$. A subset A is also *α–closed* if and only if $A = Cl_\alpha(A)$. A set U in a topological space (X, τ) is a *α-neighborhood* [7] of a point x if U contains an α–open set V such that $x \in V$.

Remark 1.1.

(i) It is shown in [7] that $\alpha O(X,\tau)$ is a topology on X and $\tau \subset \alpha O(X,\tau)$.

(ii) Clearly open sets are α–open but one easily finds in the real line with the usual topology α–open sets that are not open.

Lemma 1.2. Let A be a subset of a topological space (X,τ).

(1) $Cl_\alpha(A) = \cap\{F \in \alpha C(X,\tau) \mid A \subset F\}$.

(2) $Cl_\alpha(A)$ is α–closed, that is $Cl_\alpha(Cl_\alpha(A)) = Cl_\alpha(A)$.

Definition 1. Let A be a subset of a topological space (X,τ). A subset $\Lambda_\alpha(A)$ is defined as follows: $\Lambda_\alpha(A) = \cap\{O \in \alpha O(X,\tau) \mid A \subset O\}$ and A is called a Λ_α*-set* if $A = \Lambda_\alpha(A)$ [1].

Definition 2. Let A be a subset of a topological space (X,τ).

(i) A is called a (Λ,α)–closed set [1] if $A = T \cap C$, where T is a Λ_α–set and C is an α–closed set. The complement of a (Λ,α)–closed set is called (Λ,α)–open. We denoted the collection of all (Λ,α)–open sets (respectively (Λ,α)–closed sets) by $\Lambda_\alpha O(X,\tau)$ (respectively by $\Lambda_\alpha C(X,\tau)$).

(ii) A point $x \in X$ is called a (Λ,α)–cluster point of A [1] if for every (Λ,α)–open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all (Λ,α)–cluster points of A is called the (Λ,α)–closure of A and is denoted by $A^{(\Lambda,\alpha)}$.

Lemma 1.3 Let A and B be subsets of a topological space (X,τ). For the (Λ,α)–closure, the following properties hold [1].

(1) $A \subset A^{(\Lambda,\alpha)}$.

(2) $A^{(\Lambda,\alpha)} = \cap\{F \mid A \subset F \text{ and } F \text{ is } (\Lambda,\alpha)\text{–closed}\}$.

(3) If $A \subset B$, then $A^{(\Lambda,\alpha)} \subset B^{(\Lambda,\alpha)}$.

(4) A is (Λ,α)–closed if and only if $A = A^{(\Lambda,\alpha)}$.

(5) $A^{(\Lambda,\alpha)}$ is (Λ,α)–closed.

2. Sober Λ_α-$\mathbf{R_0}$ Topological Spaces. A. S. Davis [2] introduced the notion of R_0–axiom which in some aspects is more natural than the T_0–axiom. In this section we introduce the concept of sober Λ_α–R_0 topological space and we show that sober Λ_α–R_0 and R_0 are independent of each other.

Lemma 2.1. Let A be a subset of a space X. Then the following hold.

(1) If A is α–closed, then A is (Λ,α)–closed.

(2) If A is (Λ,α)–closed, then $A = \Lambda_\alpha(A) \cap A^{(\Lambda,\alpha)}$.

(3) If A_i is (Λ,α)–closed for each $i \in I$, then $\cap_{i\in I} A_i$ is (Λ,α)–closed.

(4) If A_i is (Λ,α)–open for each $i \in I$, then $\cup_{i\in I} A_i$ is (Λ,α)–open.

Proof. (1) It is sufficient to observe that $A = X \cap A$ where the whole set X is a Λ_α-set.

(2) Let A be (Λ, α)–closed, then there exists a Λ_α-set T and a α–closed set C such that $A = T \cap C$. By $A \subset T$, we have $A \subset \Lambda_\alpha(A) \subset \Lambda_\alpha(T) = T$, and also by $A \subset C$, $A \subset A^{(\Lambda,\alpha)} \subset C^{(\Lambda,\alpha)} = C$. Now, $A \subset \Lambda_\alpha(A) \cap A^{(\Lambda,\alpha)} \subset T \cap C = A$. Hence, $A = \Lambda_\alpha(A) \cap A^{(\Lambda,\alpha)}$.

(3) Suppose that A_i is (Λ, α)–closed for each $i \in I$. Then, for each $i \in I$ there exists a Λ_α–set T_i and a α–closed set C_i such that $A_i = T_i \cap C_i$. Now, $\bigcap_{i \in I} A_i = \bigcap_{i \in I}(T_i \cap C_i) = (\bigcap_{i \in I} T_i) \cap (\bigcap_{i \in I} C_i)$. By Lemma 2.4 of [1], $\bigcap_{i \in I} T_i$ is a Λ_α-set and $\bigcap_{i \in I} C_i$ is α–closed. This shows that $\bigcap_{i \in I} A_i$ is (Λ, α)–closed.

(4) Follows from (3).

Definition 3. Let (X, τ) be a topological space, $A \subset X$ and $x \in X$. Then

(i) The Λ_α–kernel of A, denoted by $\Lambda_\alpha Ker(A)$, is defined to be the set $\Lambda_\alpha Ker(A) = \cap\{G \in \Lambda_\alpha O(X, \tau) \mid A \subset G\}$.

(ii) $< x >= \{x\}^{(\Lambda,\alpha)} \cap \Lambda_\alpha Ker(\{x\})$.

Lemma 2.2. Let (X, τ) be a topological space and $x \in X$. Then
$\Lambda_\alpha Ker(A) = \{x \in X \mid \{x\}^{(\Lambda,\alpha)} \cap A \neq \emptyset\}$.

Proof. Let $x \in \Lambda_\alpha Ker(A)$ and suppose $\{x\}^{(\Lambda,\alpha)} \cap A = \emptyset$. Hence, $x \notin X - \{x\}^{(\Lambda,\alpha)}$ which is a (Λ, α)–open set containing A. This is absurd, since $x \in \Lambda_\alpha Ker(A)$. Consequently, $\{x\}^{(\Lambda,\alpha)} \cap A \neq \emptyset$. Next, let x such that $\{x\}^{(\Lambda,\alpha)} \cap A \neq \emptyset$ and suppose that $x \notin \Lambda_\alpha Ker(A)$. Then, there exists a (Λ, α)–open set D containing A and $x \notin D$. Let $y \in \{x\}^{(\Lambda,\alpha)} \cap A$. Hence, D is a (Λ, α)-neighborhood of y which does not contain x. But this contradicts $x \in Ker_\alpha(A)$ and the claim follows.

Lemma 2.3. If $A, B \subset X$, then

(1) $A \subset B$ implies $\Lambda_\alpha Ker(A) \subset \Lambda_\alpha Ker(B)$.

(2) $\Lambda_\alpha Ker(\Lambda_\alpha Ker(A)) = \Lambda_\alpha Ker(A)$.

Lemma 2.4. Let (X, τ) be a topological space and $x, y \in X$. Then $y \in \Lambda_\alpha Ker(\{x\})$ if and only if $x \in \{y\}^{(\Lambda,\alpha)}$.

Proof. Let $y \notin \Lambda_\alpha Ker(\{x\})$. Then there exists a (Λ, α)–open set V containing x such that $y \notin V$. Hence, $x \notin \{y\}^{(\Lambda,\alpha)}$. The converse is similarly shown.

A subset B_x of a topological space X is said to be (Λ, α)-*neighborhood* of a point $x \in X$ if there exists a (Λ, α)–open set U such that $x \in U \subset B_x$.

Lemma 2.5. A subset of a topological space X is (Λ,α)–open in X if and only if it contains a (Λ,α)–neighborhood of each of its points.

Proposition 2.6. If (X,τ) is a topological space and $A \subset X$. Then
(1) $\Lambda_\alpha Ker(A) = \{x \in X/\{x\}^{(\Lambda,\alpha)} \cap A \neq \emptyset\}$.
(2) For each $x \in X$, $\Lambda_\alpha Ker(< x >) = \Lambda_\alpha Ker(\{x\})$.
(3) For each $x \in X$, $\{< x >\}^{(\Lambda,\alpha)} = \{x\}^{(\Lambda,\alpha)}$.
(4) For each (Λ,α)–open set $U \subset X$, if $x \in U$ then $< x > \subset U$.
(5) For each (Λ,α)–closed set $F \subset X$, if $x \in F$ then $< x > \subset F$.

Proof. (1) Let $x \in \Lambda_\alpha Ker(A)$ and suppose $\{x\}^{(\Lambda,\alpha)} \cap A = \emptyset$. Then, $x \notin X \backslash \{x\}^{(\Lambda,\alpha)}$ which is a (Λ,α)–open set containing A. This is impossible, since $x \in \Lambda_\alpha Ker(A)$. Consequently, $\{x\}^{(\Lambda,\alpha)} \cap A \neq \emptyset$. Next, let $x \in X$ such that $\{x\}^{(\Lambda,\alpha)} \cap A \neq \emptyset$ and suppose that $x \notin \Lambda_\alpha Ker(A)$. Then, there exists a (Λ,α)–open set U containing A and $x \notin U$. Let $y \in \{x\}^{(\Lambda,\alpha)} \cap A$. Hence, U is a (Λ,α)-neighborhood of y which does not contain x. But this contradicts $x \in \Lambda_\alpha Ker(A)$.

(2) Follows easily from Definition 2 and Lemma 2.4.

(3) The proof is quite similar to that of (2).

(4) Since $x \in U$ and U is a (Λ,α)–open set, we have that $\Lambda_\alpha Ker(\{x\}) \subset U$. Hence, $< x > \subset U$.

(5) Since $x \in F$ and F is a (Λ,α)–closed set, we have that $< x > = \{x\}^{(\Lambda,\alpha)} \cap \Lambda_\alpha Ker(\{x\}) \subset \{x\}^{(\Lambda,\alpha)} \subset F$.

Lemma 2.7. The following statements are equivalent for any points x and y in a topological space (X,τ).
(1) $\Lambda_\alpha Ker(\{x\}) \neq \Lambda_\alpha Ker(\{y\})$.
(2) $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$.

Proof. (1) $\rightarrow$ (2): Suppose that $\Lambda_\alpha Ker(\{x\}) \neq \Lambda_\alpha Ker(\{y\})$, then there exists a point z in X such that $z \in \Lambda_\alpha Ker(\{x\})$ and $z \notin \Lambda_\alpha Ker(\{y\})$. From $z \in \Lambda_\alpha Ker(\{x\})$ it follows that $\{x\} \cap \{z\}^{(\Lambda,\alpha)} \neq \emptyset$ which implies $x \in \{z\}^{(\Lambda,\alpha)}$. By $z \notin \Lambda_\alpha Ker(\{y\})$, we have $\{y\} \cap \{z\}^{(\Lambda,\alpha)} = \emptyset$. Since $x \in \{z\}^{(\Lambda,\alpha)}$, $\{x\}^{(\Lambda,\alpha)} \subset \{z\}^{(\Lambda,\alpha)}$ and $\{y\} \cap \{x\}^{(\Lambda,\alpha)} = \emptyset$. Therefore, it follows that $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$. Now $\Lambda_\alpha Ker(\{x\}) \neq \Lambda_\alpha Ker(\{y\})$ implies that $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$.

(2) $\rightarrow$ (1): Suppose that $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$. Then there exists a point z in X such that $z \in \{x\}^{(\Lambda,\alpha)}$ and $z \notin \{y\}^{(\Lambda,\alpha)}$. It follows that there exists a (Λ,α)–open set containing z and therefore x but not y, namely, $y \notin \Lambda_\alpha Ker(\{x\})$ and thus, $\Lambda_\alpha Ker(\{x\}) \neq \Lambda_\alpha Ker(\{y\})$.

Definition 4. A topological space (X,τ) is said to be sober Λ_α–R_0 if

$\cap_{x\in X}\{x\}^{(\Lambda,\alpha)}=\emptyset$.

Theorem 2.8. A topological space (X,τ) is sober Λ_α–R_0 if and only if $\Lambda_\alpha Ker(\{x\})\neq X$ for every $x\in X$.

Proof. Suppose that the space (X,τ) is sober Λ_α–R_0. Assume that there is a point y in X such that $\Lambda_\alpha Ker(\{y\})=X$. Then $y\notin O$ which O is some proper (Λ,α)–open subset of X. This implies that $y\in\cap_{x\in X}\{x\}^{(\Lambda,\alpha)}$. But this is a contradiction. Now assume that $\Lambda_\alpha Ker(\{x\})\neq X$ for every $x\in X$. If there exists a point y in X such that $y\in\cap_{x\in X}\{x\}^{(\Lambda,\alpha)}$, then every (Λ,α)–open set containing y must contain every point of X. This implies that the space X is the unique (Λ,α)–open set containing y. Hence, $\Lambda_\alpha Ker(\{y\})=X$ which is a contradiction. Therefore, (X,τ) is sober Λ_α–R_0.

Recall that a topological space (X,τ) is said to be R_0 [2] if every open set contains the closure of each of its singletons.

Example 2.9. Let (X,τ) be a topological space such that $X=\{a,b,c\}$ and $\tau=\{\emptyset,\{a\},X\}$. Observe that (X,τ) is sober Λ_α–R_0, but it is not R_0.

Example 2.10. Let (X,τ) with $\tau=\{\emptyset,X\}$. Clearly (X,τ) is not sober Λ_α–R_0, but it is R_0.

Examples 2.9 and 2.10 show that sober Λ_α–R_0 and R_0 are independent of each other.

Definition 5. A function $f\colon X\to Y$ is always called Λ_α–closed if the image of every (Λ,α)–closed subset of X is (Λ,α)–closed in Y.

Theorem 2.11. If $f\colon X\to Y$ is an injective always Λ_α–closed function and X is sober Λ_α–R_0, then Y is sober α–R_0.

Proof. Straightforward.

Theorem 2.12. If the topological space X is sober Λ_α–R_0 and Y is any topological space, then the product $X\times Y$ is sober Λ_α–R_0.

Proof. By showing that $\cap_{(x,y)\in X\times Y}\{x,y\}^{(\Lambda,\alpha)}=\emptyset$ we are done. We have $\cap_{(x,y)\in X\times Y}\{x,y\}^{(\Lambda,\alpha)}\subseteq\cap_{(x,y)\in X\times Y}\{x\}^{(\Lambda,\alpha)}\times\{y\}^{(\Lambda,\alpha)}=\cap_{x\in X}\{x\}^{(\Lambda,\alpha)}\times\cap_{y\in Y}\{y\}^{(\Lambda,\alpha)}\subseteq\emptyset\times Y=\emptyset$.

3. Λ_α–$\mathbf{R_0}$ Topological Spaces. In this section (Λ,α)–open sets and (Λ,α)–closure operator are used to define a new separation axiom analogous to R_0- axiom and we obtain several characterizations of it.

Definition 6. A topological space (X,τ) is said to be a Λ_α–R_0 space if every (Λ,α)–open set contains the (Λ,α)–closure of each of its singletons.

The next result will give the characterization of the Λ_α-R$_0$ space in terms of the (Λ,α)–closure of points.

Theorem 3.1. For a topological space (X,τ), the following properties are equivalent.

(1) (X,τ) is a Λ_α–R_0 space.
(2) For any $F \in \Lambda_\alpha C(X,\tau), x \notin F$ implies that there exist $U \in \Lambda_\alpha O(X,\tau)$, such that $F \subset U$ and $x \notin U$.
(3) For any $F \in \Lambda_\alpha C(X,\tau), x \notin F$ implies $F \cap \{x\}^{(\Lambda,\alpha)} = \emptyset$.
(4) For any distinct points x and y of X, either $\{x\}^{(\Lambda,\alpha)} = \{y\}^{(\Lambda,\alpha)}$ or $\{x\}^{(\Lambda,\alpha)} \cap \{y\}^{(\Lambda,\alpha)} = \emptyset$.

Proof. (1) $\rightarrow$ (2): Let $F \in \Lambda_\alpha C(X,\tau)$ and $x \notin F$. Then by (1) $\{x\}^{(\Lambda,\alpha)} \subset X \backslash F$. Set $U = X\backslash\{x\}^{(\Lambda,\alpha)}$, then $U \in \Lambda_\alpha O(X,\tau)$, $F \subset U$ and $x \notin U$.

(2) $\rightarrow$ (3): Let $F \in \Lambda_\alpha C(X,\tau)$ and $x \notin F$. There exists $U \in \Lambda_\alpha O(X,\tau)$ such that $F \subset U$ and $x \notin U$. Since $U \in \Lambda_\alpha O(X,\tau), U \cap \{x\}^{(\Lambda,\alpha)} = \emptyset$ and $F \cap \{x\}^{(\Lambda,\alpha)} = \emptyset$.

(3) $\rightarrow$ (4): Assume that $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$ for distinct points $x, y \in X$. There exists $z \in \{x\}^{(\Lambda,\alpha)}$ such that $z \notin \{y\}^{(\Lambda,\alpha)}$ (or $z \in \{y\}^{(\Lambda,\alpha)}$ such that $z \notin \{x\}^{(\Lambda,\alpha)}$). There exists $V \in \Lambda_\alpha O(X,\tau)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin \{y\}^{(\Lambda,\alpha)}$. By (3), we obtain $\{x\}^{(\Lambda,\alpha)} \cap \{y\}^{(\Lambda,\alpha)} = \emptyset$. The proof for otherwise case is similar.

(4) $\rightarrow$ (1): Let $V \in \Lambda_\alpha O(X,\tau)$ and $x \in V$. For each $y \notin V, x \neq y$ and $x \notin \{y\}^{(\Lambda,\alpha)}$. This shows that $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$. By (4), $\{x\}^{(\Lambda,\alpha)} \cap \{y\}^{(\Lambda,\alpha)} = \emptyset$ for each $y \in X\backslash V$ and hence,

$$\{x\}^{(\Lambda,\alpha)} \cap \left(\bigcup_{y \in X\backslash V} \{y\}^{(\Lambda,\alpha)} \right) = \emptyset.$$

On the other hand, since $V \in \Lambda_\alpha O(X,\tau)$ and $y \in X\backslash V$, we have $\{y\}^{(\Lambda,\alpha)} \subset X\backslash V$. Therefore,

$$X\backslash V = \bigcup_{y \in X\backslash V} \{y\}^{(\Lambda,\alpha)}.$$

Therefore, we obtain $(X\backslash V) \cap \{x\}^{(\Lambda,\alpha)} = \emptyset$ and $\{x\}^{(\Lambda,\alpha)} \subset V$. Hence, (X,τ) is a Λ_α–R_0 space.

Corollary 3.2. A topological space (X,τ) is a Λ_α–R_0 space if and only if for any x and y in X, $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$ implies $\{x\}^{(\Lambda,\alpha)} \cap \{y\}^{(\Lambda,\alpha)} = \emptyset$.

Proposition 3.3. A topological space (X,τ) is Λ_α–R_0 if and only if for any points x and y and X, $\Lambda_\alpha Ker(\{x\}) \neq \Lambda_\alpha Ker(\{y\})$ implies $\Lambda_\alpha Ker(\{x\}) \cap \Lambda_\alpha Ker(\{y\}) = \emptyset$.

Proof. Suppose that (X,τ) is a Λ_α–R_0 space. Thus by Lemma 2.7, for any points x and y in X if $\Lambda_\alpha Ker(\{x\}) \neq \Lambda_\alpha Ker(\{y\})$ then $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$. Now we prove that $\Lambda_\alpha Ker(\{x\}) \cap \Lambda_\alpha Ker(\{y\}) = \emptyset$. Assume that $z \in \Lambda_\alpha Ker(\{x\}) \cap \Lambda_\alpha Ker(\{y\})$. By $z \in \Lambda_\alpha Ker(\{x\})$, it follows that $x \in \{z\}^{(\Lambda,\alpha)}$. Since $x \in \{x\}^{(\Lambda,\alpha)}$, by Corollary 3.2 $\{x\}^{(\Lambda,\alpha)} = \{z\}^{(\Lambda,\alpha)}$. Similarly, we have $\{y\}^{(\Lambda,\alpha)} = \{z\}^{(\Lambda,\alpha)} = \{x\}^{(\Lambda,\alpha)}$. This is a contradiction. Therefore, we have $\Lambda_\alpha Ker(\{x\}) \cap \Lambda_\alpha Ker(\{y\}) = \emptyset$.

Conversely, let (X,τ) be a topological space such that for any points x and y in X, $\Lambda_\alpha Ker(\{x\}) \neq \Lambda_\alpha Ker(\{y\})$ implies $\Lambda_\alpha Ker(\{x\}) \cap \Lambda_\alpha Ker(\{y\}) = \emptyset$. If $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$, then by Lemma 2.7, $\Lambda_\alpha Ker(\{x\}) \neq \Lambda_\alpha Ker(\{y\})$. Therefore, $\Lambda_\alpha Ker(\{x\}) \cap \Lambda_\alpha Ker(\{y\}) = \emptyset$ which implies $\{x\}^{(\Lambda,\alpha)} \cap \{y\}^{(\Lambda,\alpha)} = \emptyset$. Because $z \in \{x\}^{(\Lambda,\alpha)}$ implies that $x \in \Lambda_\alpha Ker(\{z\})$, $\Lambda_\alpha Ker(\{x\}) \cap \Lambda_\alpha Ker(\{z\}) \neq \emptyset$. By hypothesis, we therefore have $\Lambda_\alpha Ker(\{x\}) = \Lambda_\alpha Ker(\{z\})$. Then $z \in \{x\}^{(\Lambda,\alpha)} \cap \{y\}^{(\Lambda,\alpha)}$ implies that $\{x\}^{(\Lambda,\alpha)} = \{z\}^{(\Lambda,\alpha)} = \{y\}^{(\Lambda,\alpha)}$. This is a contradiction. Therefore, $\{x\}^{(\Lambda,\alpha)} \cap \{y\}^{(\Lambda,\alpha)} = \emptyset$ and by Corollary 3.2, (X,τ) is a Λ_α–R_0 space.

Proposition 3.4. For a topological space (X,τ), the following properties are equivalent.

(1) (X,τ) is a Λ_α–R_0 space.
(2) For any nonempty set A and $G \in \Lambda_\alpha O(X,\tau)$ such that $A \cap G \neq \emptyset$, there exists $F \in \Lambda_\alpha C(X,\tau)$ such that $A \cap F \neq \emptyset$ and $F \subset G$.
(3) Any $G \in \Lambda_\alpha O(X,\tau)$, $G = \cup\{F \in \Lambda_\alpha C(X,\tau) \mid F \subset G\}$.
(4) Any $F \in \Lambda_\alpha C(X,\tau)$, $F = \cap\{G \in \Lambda_\alpha O(X,\tau) \mid F \subset G\}$ (i.e., $F = \Lambda_\alpha Ker(F)$).
(5) For any $x \in X$, $\{x\}^{(\Lambda,\alpha)} \subset \Lambda_\alpha Ker(\{x\})$.

Proof. (1) $\to$ (2): Let A be a nonempty set of X and $G \in \Lambda_\alpha O(X,\tau)$ such that $A \cap G \neq \emptyset$. There exists $x \in A \cap G$. Since $x \in G \in \Lambda_\alpha O(X,\tau)$, $\{x\}^{(\Lambda,\alpha)} \subset G$. Set $F = \{x\}^{(\Lambda,\alpha)}$, then $F \in \Lambda_\alpha C(X,\tau), F \subset G$ and $A \cap F \neq \emptyset$.

(2) $\to$ (3): Let $G \in \Lambda_\alpha O(X,\tau)$, then $G \supset \cup\{F \in \Lambda_\alpha C(X,\tau) \mid F \subset G\}$. Let x be any point of G. There exists $F \in \Lambda_\alpha C(X,\tau)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \cup\{F \in \Lambda_\alpha C(X,\tau) \mid F \subset G\}$ and hence, $G = \cup\{F \in \Lambda_\alpha C(X,\tau) \mid F \subset G\}$.

(3) $\to$ (4): This is obvious.

(4) $\rightarrow$ (5): It follows from (4) and Lemma 1.3.

(5) $\rightarrow$ (1): Let $G \in \Lambda_\alpha O(X,\tau)$ and $x \in G$. Let $y \in \Lambda_\alpha Ker(\{x\})$, then $x \in \{y\}^{(\Lambda,\alpha)}$ and $y \in G$. This implies that $\Lambda_\alpha Ker(\{x\}) \subset G$. Therefore, we obtain $x \in \{x\}^{(\Lambda,\alpha)} \subset \Lambda_\alpha Ker(\{x\}) \subset G$. This shows that (X,τ) is a Λ_α–R_0 space.

Corollary 3.5. For a topological space (X,τ), the following properties are equivalent.
(1) (X,τ) is a Λ_α–R_0 space.
(2) $\{x\}^{(\Lambda,\alpha)} = \Lambda_\alpha Ker(\{x\})$ for all $x \in X$.

Proof. (1) $\rightarrow$ (2): Suppose that (X,τ) is a Λ_α–R_0 space. By Proposition 3.4, $\{x\}^{(\Lambda,\alpha)} \subset \Lambda_\alpha Ker(\{x\})$ for each $x \in X$. Let $y \in \Lambda_\alpha Ker(\{x\})$, then $\{x\}^{(\Lambda,\alpha)} \cap \{y\}^{(\Lambda,\alpha)} \neq \emptyset$. By Corollary 3.2 we obtain $\{x\}^{(\Lambda,\alpha)} = \{y\}^{(\Lambda,\alpha)}$. Therefore, $y \in \{x\}^{(\Lambda,\alpha)}$ and hence, $\Lambda_\alpha Ker(\{x\}) \subset \{x\}^{(\Lambda,\alpha)}$. This shows that $\{x\}^{(\Lambda,\alpha)} = \Lambda_\alpha Ker(\{x\})$.

(2) $\rightarrow$ (1): Proposition 3.4.

Corollary 3.6. Let X be a Λ_α–R_0 topological space. For any $x \in X$ if $< x >= \{x\}$, then $\{x\}^{(\Lambda,\alpha)} = \{x\}$.

Proof. It is a consequence of Corollary 3.5.

Proposition 3.7. For a topological space (X,τ), the following properties are equivalent.
(1) (X,τ) is a Λ_α–R_0 space.
(2) $x \in \{y\}^{(\Lambda,\alpha)}$ if and only if $y \in \{x\}^{(\Lambda,\alpha)}$.

Proof. (1) $\rightarrow$ (2): Let X be Λ_α–R_0. Let $x \in \{y\}^{(\Lambda,\alpha)}$ and U be any (Λ,α)–open set such that $y \in U$. Hence, $\Lambda_\alpha Ker(\{y\}) \subset U$. Since $x \in \{y\}^{(\Lambda,\alpha)}$ and (X,τ) is Λ_α–R_0, by Corollary 3.5, $x \in \Lambda_\alpha Ker(\{y\}) \subset U$. Therefore, every (Λ,α)–open set which contains y contains x. Hence, $y \in \{x\}^{(\Lambda,\alpha)}$.

(2) $\rightarrow$ (1): Let U be a (Λ,α)–open set and $x \in U$. If $y \notin U$, then $x \notin \{y\}^{(\Lambda,\alpha)}$ and hence, $y \notin \{x\}^{(\Lambda,\alpha)}$. This implies that $\{x\}^{(\Lambda,\alpha)} \subset U$. Hence, (X,τ) is a Λ_α–R_0 space.

Proposition 3.8. For a topological space (X,τ), the following properties are equivalent.
(1) X is a Λ_α–R_0 space.
(2) $< x >= \{x\}^{(\Lambda,\alpha)}$ for each $x \in X$.
(3) $< x >$ is (Λ,α)–closed for each $x \in X$.

Proof. (1) $\to$ (2): By Corollary 3.5, $\{x\}^{(\Lambda,\alpha)} = \Lambda_\alpha Ker(\{x\})$ for each $x \in X$. Hence, $\{x\}^{(\Lambda,\alpha)} = \{x\}^{(\Lambda,\alpha)} \cap \Lambda_\alpha Ker(\{x\}) =< x >$.

(2) $\to$ (1): Since $\{x\}^{(\Lambda,\alpha)} =< x >$ for each $x \in X$, we have $\{x\}^{(\Lambda,\alpha)} \subset \Lambda_\alpha Ker(\{x\})$. By Proposition 3.4(5), X is Λ_α–R_0.

(2) $\longleftrightarrow$ (3): It is consequence of Lemma 2.1.

4. Λ_α–$\mathbf{R_1}$ Topological Spaces and a Question. A. S. Davis [2] also introduced the notion of R_1 topological spaces which is strictly weaker than T_2. Now we offer a new generalization of R_1 by utilizing the notions of (Λ,α)–open sets and (Λ,α)–closure operator.

Definition 7. A topological space (X,τ) is said to be Λ_α–R_1 if for x, y in X with $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$, there exists disjoint (Λ,α)–open sets U and V such that $\{x\}^{(\Lambda,\alpha)}$ is a subset of U and $\{y\}^{(\Lambda,\alpha)}$ is a subset of V.

Proposition 4.1. If (X,τ) is Λ_α–R_1, then (X,τ) is Λ_α–R_0.

Proof. Let U be (Λ,α)–open and let $x \in U$. If $y \notin U$, then since $x \notin \{y\}^{(\Lambda,\alpha)}$, $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$ and there exists a (Λ,α)–open V_y such that $\{y\}^{(\Lambda,\alpha)} \subset V_y$ and $x \notin V_y$, which implies $y \notin \{x\}^{(\Lambda,\alpha)}$. Thus, $\{x\}^{(\Lambda,\alpha)} \subset U$. Hence, (X,τ) is Λ_α–R_0.

Recall that a topological space (X,τ) is said to be R_1 [2] if for x, y in X with $Cl(\{x\}) \neq Cl(\{y\})$, there exists disjoint open sets U and V such that $Cl(\{x\})$ is a subset of U and $Cl(\{y\})$ is a subset of V.

Example 4.2. Let (X,τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Clearly, the family of all (Λ,α)–closed sets consists of

$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. We have the following for the space (X,τ).

1) (X,τ) is Λ_α–R_0, but it is not R_0.
2) (X,τ) is Λ_α–R_1, but it is not R_1.

Question. Characterize Λ_α–R_1 spaces. Is there any example showing that a topological space is Λ_α–R_0 but not Λ_α–R_1?

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References

1. M. Caldas, D. N. Georgiou, and S. Jafari, "Study of α–open Sets and the Related Notions in Topological Spaces (submitted).

2. A. S. Davis, "Indexed Systems of Neighborhoods for General Topological Spaces," *American Mathematical Monthly,* 68 (1961), 886–893.

3. K. K. Dube, "A Note on R_0 Topological Spaces," *Math. Vesnik,* 11 (1974), 203–208.

4. D. W. Hall, S. K. Murphy, and J. Rozycki, "On Spaces Which are Essentially T_1, *J. Austr. Math. Soc.,* 12 (1974), 451–455.

5. H. Herlich, "A Concept of Nearness," *Gen. Topol. Appl.,* 4 (1974), 191–212.

6. S. A. Naimpally, "On R_0-Topological Spaces," *Ann. Univ. Sci. Budapest Eotvos,* Sect. Math. 10 (1967), 53–54.

7. O. Njåstad, "On Some Classes of Nearly Open Sets," *Pacific Journal of Mathematics,* 15 (1965), 961–970.

8. N. A. Shanin, "On Separation in Topological Spaces," *Dokl. Akad. Nauk. SSSR,* 38 (1943), 110–113.

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