POWER SETS OF POLYNOMIALS IN FABER REGIONS

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Abstract. The representation of the regular functions by power sets of polynomials of the single complex variable z in Faber regions is the subject of the present work. Different kinds of order and type of these sets are investigated.

1. Preliminaries. The transformation

$$z = \phi(t) = t + \sum_{n=0}^{\infty} \alpha_n t^{-n} = t + M\left(\frac{1}{t}\right), \tag{1.1}$$

was introduced and studied by Faber [3], and is called the Faber transformation. Such a transformation is conformal for $T_0 < |t| < \infty$. If γ is a fixed number greater than T_0, then the circle $|t| = \gamma$ is mapped onto the simple regular closed curve C in the z–plane. When $T_0 < r < \infty$, the circles $|t| = r$ are mapped onto the simple closed regular curves C_r such that when $r_1 < r_2$ the curve C_{r_1} lies totally inside the curve C_{r_2}. The set of curves (C_r) are called Faber curves and so the typical representative curve C is the Faber curve C_γ. The regions $D(C)$ and $\bar{D}(C)$ which denote the set of points in the interior of C and its closure are called the Faber regions. Let us investigate the expression $\frac{d\zeta}{\zeta - z}$ under the integral sign in Cauchy's formula where ζ is the integration variable on the z–plane. Substituting $\zeta = \phi(t)$, we get

$$\frac{d\zeta}{\zeta - z} = \frac{\phi^{'}(t)dt}{\phi(t) - z}. \tag{1.2}$$

Consider the following function of t:

$$G(z,t) = \frac{t\phi^{'}(t)dt}{\phi(t) - z} = \sum_{n=0}^{\infty} P_n(z) t^{-n} \tag{1.3}$$

where $P_n(z)$ is a polynomial of degree n. The simple monic set $\{P_n(z)\}$ of polynomials is called the set of Faber polynomials associated with the transformation

$\phi(t)$. The function $G(z,t)$ is called the generating function of Faber polynomials which is regular at the point $t=\infty$ for any z, $G(z,\infty)$ and the function $G(z,t)$ can have a singularity only at points t satisfying $\phi(t)=z$.

Since the transformation $z=\phi(t)$ is conformal in $|z|>T_0$, then it has an inverse of the form

$$t=\psi(z)=z+\sum_{n=0}^{\infty}\beta_n z^{-n}, \tag{1.4}$$

which is still a Faber transformation, being conformal in $|t|>\bar{T}$.

It has been proved by Newns [6], that

$$P_n(z)=\{\psi(z)\}^n+\frac{1}{2\pi i}\int_{C_{T'}}\frac{\{\psi(\zeta)\}^n d\zeta}{\zeta-z}; \quad n\geq 0, \tag{1.5}$$

where $T^{'}$ is any number greater than T_0 and $z\in C_r$, where $r>T^{'}$.

The following statement is a consequence of relation (1.5) and has been stated by Ullman [9] that

$$P_n(z) \text{ is the polynomial part of } \{\psi(z)\}^n. \tag{1.6}$$

The inverse transformation $\psi(t)$ has the following relations

$$\frac{t\psi^{'}(t)}{\psi(t)-z}=\sum_{n=0}^{\infty}\bar{P}_n(z)t^{-n}, \tag{1.7}$$

$$\bar{P}_n(z)=\{\phi(z)\}^n+\frac{1}{2\pi i}\int_{|t|=T'}\frac{\{\phi(t)\}^n dt}{t-z} \tag{1.8}$$

and

$$\bar{P}_n(z) \;\text{ is the polynomial part of } \{\phi(z)\}^n \tag{1.9}$$

where $\{\bar{P}_n(z)\}$ is the inverse of the set $\{P_n(z)\}$.

Now, differentiating (1.3) n times with respect to z and then putting $z = 0$, we obtain

$$\frac{t\phi'(t)}{\{\phi(t)\}^{n+1}} = \sum_{k=n}^{\infty} \frac{P_k^{(n)}(0)}{n!} t^{-k} = \sum_{k=n}^{\infty} P_{k,n} t^{-k}. \tag{1.10}$$

A sequence $p_0(z)$, $p_1(z)$, $p_2(z)$, $\ldots$, $p_n(z)$, $\ldots$ of polynomials where

$$p_n(z) = \sum_k p_{n,k} z^k \tag{1.11}$$

is said to form a basic set $\{p_n(z)\}$, (c.f. [10]), if any polynomial, and in particular, the monomial z^n, admits a unique finite representation of the form

$$z^n = \sum_k \pi_{n,k} p_k(z). \tag{1.12}$$

Let $z = \phi(t)$ be a Faber transformation which is conformal in $T_0 < |t| < \infty$, where the circles $|t| = r$, $T_0 < r < \infty$, are mapped in the z–plane on the Faber curves C_r, and suppose that $\{\Pi_n(z)\}$ is the set of Faber polynomials associated with the transformation $\phi(t)$.

If the number N_n of non–zero coefficients in (1.12) is such that $\lim_{n\to\infty} N_n^{\frac{1}{n}} = 1$, the set $\{p_n(z)\}$ is called a Cannon set (c.f. [10]) of polynomials which admits the unique representation

$$\Pi_n(z) = \sum_{k=0}^{\infty} \pi_{n,k} p_k(z), \quad (n \geq 0). \tag{1.13}$$

If $f(z)$ is an analytic function, which is regular in $\bar{D}(C_r)$; $r > T_0$, then , according to Faber [3], $f(z)$ will admit the Faber series

$$f(z) = \sum_{n=0}^{\infty} c_n P_n(z) \tag{1.14}$$

in $\bar{D}(C_r)$. Introducing (1.13) in (1.14) we get the basic series associated with $f(z)$ in the form

$$f(z) \sim \sum_{n=0}^{\infty} A_n p_n(z) \tag{1.15}$$

where

$$A_n = \sum_{k=0}^{\infty} c_k \pi_{k,n}.$$

The basic series (1.15) associated with $f(z)$ is said to represent $f(z)$ in $\bar{D}(C_r)$ if it converges uniformly to $f(z)$. In this case we say that the set $\{p_n(z)\}$ represents $f(z)$ in $\bar{D}(C_r)$.

The basic set $\{p_n(z)\}$ is said to be effective in $\bar{D}(C_r)$ for the class $\bar{H}(C_\rho)$ of functions regular in $\bar{D}(C_\rho)$, where $\rho \geq r$, if the set $\{p_n(z)\}$ represents in $\bar{D}(C_r)$ every function of the class $\bar{H}(C_\rho)$, this is justified by the Cannon function $\lambda\{p_n; C_r\}$ for the Faber regions $\bar{D}(C_r)$ in the form

$$\lambda\{p_n; C_r\} = \limsup_{n\to\infty} \{\omega_n(C_r)\}^{\frac{1}{n}} \tag{1.16}$$

where

$$\omega_n(C_r) = \sum_{k=0}^{\infty} |\pi_{n,k}| M(p_k; C_r) \tag{1.17}$$

and

$$M(p_k; C_r) = \max_{z \in C_r} |p_k(z)|.$$

According to Newns [5] the necessary and sufficient condition for the Cannon set $\{p_n(z)\}$ to be effective in $\bar{D}(C_r)$ for the class $\bar{H}(C_\rho)$; $\rho \geq r$ is that

$$\lambda\{p_n; C_r\} = \rho. \tag{1.18}$$

Let $\{p_n(z)\}$ be a given basic set of polynomials, for which the representation (1.13) holds, and suppose further that

$$p_n(z) = \sum_k q_{n,k} P_k(z); \quad (n \geq 0). \tag{1.19}$$

If we write

$$q_n(z) = \sum_k q_{n,k} z^k, \tag{1.20}$$

then $\{q_n(z)\}$ is a basic set and we can write the set $\{p_n(z)\}$ as the product set of two sets $\{q_n(z)\}$ and $\{P_n(z)\}$ as follows:

$$\{p_n(z)\} = \{q_n(z)\}\{P_n(z)\}. \tag{1.21}$$

The set $\{q_n(z)\}$ is called the set associated with $\{p_n(z)\}$ with respect to the transformation $\phi(t)$. Nassif [4] proved that

$$\lambda\{p_n; C_r\} = \lambda\{q_n; r\} \tag{1.22}$$

where $\lambda\{q_n; r\}$ is the Cannon function of $\{q_n(z)\}$ for the region $|z| \leq r$.

Thus, to investigate the effectiveness of a given set $\{p_n(z)\}$ in the Faber region $\bar{D}(C_r)$ is equivalent to the study of the effectiveness of the associated set $\{q_n(z)\}$ in $|z| \leq r$.

The set $\{p_n(z)\}$ of polynomials in which the polynomial $p_n(z)$ is of degree n is called a simple set. Such a set for which $p_{n,n} = 1$ for all $n \geq 0$, is called a simple monic set. The simple set for which $|p_{n,n}| = 1$, for all $n \geq 0$, is called a simple absolutely monic set.

2. Power Sets of Faber Polynomials. Let $\{p_n(z)\}$ be a basic set of polynomials of the single complex variable z, where

$$p_n(z) = \sum_j p_{n,j} z^j; \quad n > 0.$$

The power set of polynomials $\{p_n^{(\mu)}(z)\}$ was given in the form

$$p_n^{(\mu)}(z) = \sum_k p_{n,k}^{(\mu)} z^k = \sum_{h_1, h_2, \ldots, h_{\mu-1}, k} p_{n,h_1} p_{h_1,h_2} \cdots p_{h_{\mu-1},k} z^k, \tag{2.1}$$

where μ is a finite positive integer.

Let $\phi(t) = t + \sum_{n=0}^{\infty} \alpha_n t^{-n}$ be a Faber transformation with inverse

$$\psi(t) = t + \sum_{n=0}^{\infty} \beta_n t^{-n}.$$

The composite transformation

$$\phi_\mu(t) = \phi(\phi(\dots(\phi(t))\dots)), \tag{2.2}$$

is still a Faber transformation of the form

$$\phi_\mu(t) = t + \sum_{n=0}^{\infty} \alpha_n^* t^{-n}. \tag{2.3}$$

Suppose that $\{\Pi_n(z)\}$ is the set of Faber polynomials associated with the transformation $\phi(t)$.

Let $\{P_n^{(\mu)}(z)\}$ be the power set of Faber polynomials given in the form

$$\{P_n^{(\mu)}(z)\} = \{P_n(z)\}\{P_n^{(\mu-1)}(z)\} \tag{2.4}$$

where

$$P_n^{(\mu)}(z) = \sum_{h_1=0}^{n} \sum_{h_2=0}^{h_1} \cdots \sum_{h_{\mu-1}=0}^{h_{\mu-2}} \sum_{k=0}^{h_{\mu-1}} P_{n,h_1} P_{h_1,h_2} \dots P_{h_{\mu-1},k} z^k,$$

$$P_n(z) = z^n + \sum_{k=0}^{n-1} P_{n,k} z^k, \tag{2.5}$$

$$\bar{P}_n(z) = z^n + \sum_{k=0}^{n-1} \bar{P}_{n,k} z^k, \tag{2.6}$$

$$P_n^{(\mu)}(z) = z^n + \sum_{k=0}^{n-1} P_{n,k}^{(\mu)} z^k \tag{2.7}$$

and

$$P_{n,k}^{(\mu)} = \sum_i P_{n,i} P_{i,k}^{(\mu-1)}; \quad (n \geq 0). \tag{2.8}$$

The following result concerning Faber power sets of polynomials is proved by using mathematical induction.

Theorem 2.1. The power set as given in (2.4) is the set of Faber polynomials associated with the composite transformation (2.2).

Proof. From (2.4) and (2.9) we have

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n^{(\mu)} t^{-n} &= \sum_{n=0}^{\infty} \left[\sum_{h_1=0}^{n} \sum_{h_2=0}^{h_1} \cdots \sum_{h_{\mu-1}=0}^{h_{\mu-2}} \sum_{k=0}^{h_{\mu-1}} P_{n,h_1} P_{h_1,h_2} \ldots P_{h_{\mu-1},k} z^k \right] t^{-n} \\
&= \sum_{n=0}^{\infty} \left[\sum_{h_1=0}^{n} P_{n,h_1} t^{-n} \right] \sum_{h_2=0}^{h_1} \cdots \sum_{h_{\mu-1}=0}^{h_{\mu-2}} \sum_{k=0}^{h_{\mu-1}} P_{h_1,h_2} \ldots P_{h_{\mu-1},k} z^k \\
&= \sum_{h_1=0}^{\infty} \left[\sum_{h_1=n}^{\infty} P_{n,h_1} t^{-n} \right] \sum_{h_2=0}^{h_1} \cdots \sum_{h_{\mu-1}=0}^{h_{\mu-2}} \sum_{k=0}^{h_{\mu-1}} P_{h_1,h_2} \ldots P_{h_{\mu-1},k} z^k \\
&= \frac{t\phi'(t)}{\phi(t)} \sum_{h_1=0}^{\infty} \sum_{h_2=0}^{h_1} \cdots \sum_{h_{\mu-1}=0}^{h_{\mu-2}} \sum_{k=0}^{h_{\mu-1}} P_{h_1,h_2} \ldots P_{h_{\mu-1},k} z^k \{\phi(t)\}^{-h_1} \\
&= \frac{t\phi'(t)\phi'(\phi(t))}{\phi(\phi(t))} \sum_{h_2=0}^{\infty} \sum_{h_3=0}^{h_2} \cdots \sum_{h_{\mu-1}=0}^{h_{\mu-2}} \sum_{k=0}^{h_{\mu-1}} P_{h_2,h_3} \ldots P_{h_{\mu-1},k} z^k \{\phi(\phi(t))\}^{-h_2}
\end{aligned}$$

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$$= \frac{t\phi'_\mu(t)}{\phi_\mu(t)} \sum_{k=0}^{\infty} \left\{ \frac{z}{\phi_\mu(t)} \right\}^k$$

$$= \frac{t\phi'_\mu(t)}{\phi_\mu(t) - z}.$$

That is,

$$\sum_{n=0}^{\infty} P_n^{(\mu)}(z) t^{-n} = \frac{t\phi'_\mu(t)}{\phi_\mu(t) - z}. \tag{2.9}$$

Therefore the set $\{P_n^{(\mu)}(z)\}$ is a set of Faber polynomials associated with the composite transformation (2.2), and this completes the proof of Theorem 2.1.

3. Effectiveness of Power Sets of Polynomials in Faber Regions. Let $\{p_n(z)\}$ be a simple absolutely monic set of polynomials, and $\{p_n^{(\mu)}(z)\}$ be the power set of polynomials of the form

$$\{p_n^{(\mu)}(z)\} = \{p_n(z)\}\{p_n^{(\mu-1)}(z)\}. \tag{3.1}$$

Hence, we have

$$p_n^{(\mu)}(z) = \sum_i p_{n,i}^{(\mu)} z^i = \sum_i \sum_k p_{n,i} p_{i,k}^{(\mu-1)} z^k; \quad n \geq 0.$$

Suppose that the Faber transformation

$$z = \phi(t) = t + \sum_{n=0}^{\infty} \alpha_n t^{-n} \tag{3.2}$$

is conformal in $|t| \geq T$, then the inverse transformation is such that

$$z = \psi(t) = t + \sum_{n=0}^{\infty} \beta_n t^{-n} \tag{3.3}$$

will also be conformal in $|t| \geq \bar{T}$.

The composite transformation (2.2) is conformal in $|t| \geq T_0$. Take the number γ such that

$$\gamma > (T, \bar{T}, T_0) \tag{3.4}$$

and suppose that the Faber curves C, $\bar{C}$ and C^{μ} are the respective images, in the z plane, of the circle $|t| = \gamma$ by the transformations (2.2), (3.2) and (3.3). The curve C^{μ} is said to be the power curve of C.

Now, we study effectiveness of the power set $\{P_n^{(\mu)}(z)\}$ in the Faber region $\bar{D}(C)$. For this purpose consider the following

$$G(r) = \sup_{|t|=r} |\phi(t)|, \quad r > T \tag{3.5}$$

and

$$H(r) = \sup_{|t|=r} |\psi(t)|, \quad r > T. \tag{3.6}$$

Let $\{q_n(z)\}$, $\{h_n(z)\}$ and $\{g_n(z)\}$ be the sets of polynomials associated respectively with sets $\{p_n(z)\}$, $\{p_n^{(\mu-1)}(z)\}$ and $\{p_n^{(\mu)}(z)\}$ with the respective transformations $\phi(t)$, $\phi_{\mu-1}(t)$, and $\phi_\mu(t)$.

From (1.21) one gets

$$\{p_n(z)\} = \{q_n(z)\}\{P_n(z)\}, \tag{3.7}$$

$$\{p_n^{(\mu-1)}(z)\} = \{q_n(z)\}\{P_n^{(\mu-1)}(z)\}, \tag{3.8}$$

and

$$\{p_n^{(\mu)}(z)\} = \{g_n(z)\}\{P_n^{(\mu)}(z)\}. \tag{3.9}$$

For $\mu = 2$, from (3.7) and (3.9) we get

$$\begin{aligned} \{g_n(z)\} &= \{p_n^{(2)}(z)\}\{\bar{P}_n^{(2)}(z)\} \\ &= \{p_n(z)\}\{p_n(z)\}\{\bar{P}_n(z)\}\{\bar{P}_n(z)\} \\ &= \{q_n(z)\}\{P_n(z)\}\{q_n(z)\}\{\bar{P}_n(z)\} \end{aligned} \tag{3.10}$$

and

$$\{\bar{g}_n(z)\} = \{P_n(z)\}\{\bar{q}_n(z)\}\{\bar{P}_n(z)\}\{\bar{q}_n(z)\}. \tag{3.11}$$

Suppose that the set $\{p_n(z)\}$ is effective in the Faber region $\bar{D}(C)$, then the set $\{\bar{p}_n(z)\}$ is effective in the same region (c.f. [1]), and its associated sets $\{q_n(z)\}$ and

$\{\bar{q}_n(z)\}$ are effective in $|z| \leq \gamma$. Since effectiveness of simple sets in $|z| \leq \gamma$ implies effectiveness in $|z| \leq r$ for all $r \geq \gamma$; this happens for absolutely simple monic sets, so the sets $\{q_n(z)\}$ and $\{\bar{q}_n(z)\}$ are effective in $|z| \leq r$, for all $r \geq \gamma$.

We shall use an increasing sequence $(r_j); j = 1, 2, \ldots$ of positive numbers greater than γ, i.e., $r_{j+1} > r_j$, for all $j, r_1 > \gamma$.

From (1.5), (1.8), (3.5), and (3.6) we have

$$M(P_n; r_3) < k\{H(r_3)\}^n \tag{3.12}$$

and

$$M(\bar{P}_n; \gamma) < k\{G(r)\}^n; \quad n \geq 0. \tag{3.13}$$

Write

$$G(\gamma) < r_5, \quad H(r_3) < r_4. \tag{3.14}$$

Since the sets $\{q_n(z)\}$ and $\{\bar{q}_n(z)\}$ are simple absolutely monic, then we have

$$M(q_n; r_4) < kr_7^n \tag{3.15}$$

and

$$M(\bar{q}_n; r_2) < kr_5^n. \tag{3.16}$$

By using Cauchy's inequality and the relations (3.12), (3.13), (3.14), and (3.15) we get

$$\begin{aligned} M(g_n; \gamma) &= \max_{|z|=\gamma} |g_n(z)| \\ &\leq \sum_{i_1, i_2, i_3, i_4} |q_{n,i_1}||P_{i_1,i_2}||q_{i_2,i_3}||\bar{P}_{i_3,i_4}|\gamma^{i_4} \\ &< K(n+1)^4 r_7^n \left(\frac{r_7}{r_3}\right)^n \left(\frac{r_5}{r_4}\right)^n. \end{aligned} \tag{3.17}$$

The inverse set $\{\bar{g}_n(z)\}$, also has the following relations

$$M(P_n; r_{10}) < k\{H(r_{10})\}^n, \tag{3.18}$$

$$M(\bar{P}_n; r_5) < k\{G(r_5)\}^n, \tag{3.19}$$

$$G(r_5) < r_9 \tag{3.20}$$

and

$$M(\bar{q}_n; r_4) < k r_{11}; \quad n \geq 0. \tag{3.21}$$

It follows from (3.11), (3.13), (3.20), (3.21), and by using Cauchy's inequality that

$$\begin{aligned} M(\bar{g}_n; r_4) &= \max_{|z|=r_4} |\bar{g}_n(z)| \\ &\leq \sum_{i_1,i_2,i_3,i_4} |P_{n,i_1}||\bar{q}_{i_1,i_2}||\bar{P}_{i_2,i_3}||\bar{q}_{i_3,i_4}| r_4^{i_4} \\ &< K(n+1)^4 \left(\frac{r_{11}}{r_{10}}\right)^n \left(\frac{r_9}{r_4}\right)^n \left(\frac{r_{11}}{r_5}\right)^n \{H(r_{10})\}^n. \end{aligned} \tag{3.22}$$

Inserting (3.17) and (3.22) in the Cannon sum $\omega_n(\gamma)$ of the set $\{g_n(z)\}$ we obtain

$$\begin{aligned} \omega_n(\gamma) &= \sum_k |\bar{g}_{n,k}| M(g_k; \gamma) \\ &\leq \sum_k \frac{M(\bar{g}_n; r_4)}{r_4^k} M(g_k; \gamma) \\ &< K(n+1)^8 \left(\frac{r_{11}}{r_{10}}\right)^n \left(\frac{r_9}{r_4}\right)^n \left(\frac{r_{11}}{r_5}\right)^n \left(\frac{r_7}{r_4}\right)^n \left(\frac{r_7}{r_3}\right)^n \left(\frac{r_5}{r_4}\right)^n \{H(r_{10})\}^n. \end{aligned} \tag{3.23}$$

The Cannon function $\lambda(g_n; \gamma)$ of the set $\{g_n(z)\}$ is

$$\begin{aligned} \lambda(g_n; \gamma) &= \limsup_{n\to\infty} \{\omega_n(\gamma)\}^{\frac{1}{n}} \\ &\leq K \left(\frac{r_{11}}{r_{10}}\right) \left(\frac{r_9}{r_4}\right) \left(\frac{r_{11}}{r_5}\right) \left(\frac{r_7}{r_4}\right) \left(\frac{r_7}{r_3}\right) \left(\frac{r_5}{r_4}\right) \{H(r_{10})\}. \end{aligned} \tag{3.24}$$

Since r_{11} can be taken arbitrarily close to γ, we get

$$\lambda(g_n; \gamma) \leq H(\gamma) = A. \tag{3.25}$$

Applying (1.22) we get

$$\lambda(p_n; C^2) = \lambda(g_n; \gamma) \le A$$

where $\lambda(p_n; C^2)$ is the Cannon function of set $\{p_n^{(2)}(z)\}$ for the region $\bar{D}(C^2)$.

Therefore the set $\{p_n^{(2)}(z)\}$ is effective in the region $\bar{D}(C^2)$ for the class of functions $\bar{H}(C_A^2)$.

Now, suppose that the set $\{p_n^{(\mu-1)}(z)\}$ is effective in the region $\bar{D}(C^{(\mu-1)})$ for the class of functions $\bar{H}(C_A^{\mu-1})$.

We shall prove that $\{p_n^{(\mu)}(z)\}$ is effective in the same region for the same class, when the set $\{p_n(z)\}$ is a simple absolutely monic set which is effective in the region $\bar{D}(C^\mu)$.

From (3.1), (3.8), (3.9), and (3.10) we have

$$\begin{aligned} \{g_n(z)\} &= \{p_n^{(\mu)}(z)\}\{\bar{P}_n^{(\mu)}(z)\} \\ &= \{p_n(z)\}\{p_n^{(\mu-1)}(z)\}\{\bar{P}_n^{(\mu-1)}(z)\}\{\bar{P}_n(z)\} \\ &= \{q_n(z)\}\{P_n(z)\}\{h_n(z)\}\{\bar{P}_n(z)\} \end{aligned} \tag{3.26}$$

and

$$\{\bar{g}_n(z)\} = \{P_n(z)\}\{\bar{h}_n(z)\}\{\bar{P}_n(z)\}\{\bar{q}_n(z)\}. \tag{3.27}$$

Proceeding similar to the case when $\mu = 2$ above, we see directly that the set $\{p_n^{(\mu)}(z)\}$ is effective in the region $\bar{D}(C^\mu)$ for the class of functions $\bar{H}(C_A^\mu)$.

This completes the proof of the following main result.

Theorem 3.1. Let $\{p_n(z)\}$ be a simple absolutely monic set of polynomials, effective in the Faber region $\bar{D}(C)$ where C is the Faber curve. The number $\gamma > (T, T_0)$ can be chosen such that

$$G(\gamma) > T \text{ and } G(\gamma) > T_0. \tag{3.28}$$

Then the power set $\{p_n^\mu(z)\} = \{p_n(z)\}\{p_n^{(\mu-1)}(z)\}$ will be effective in the Faber region $\bar{D}(C^\mu)$ where C^μ is the power curve of C, for the class of functions $\bar{H}(C_A^\mu)$, where

$$A = H(\gamma). \tag{3.29}$$

4. Mode of Increase of Power Sets of Polynomials. We give, for the sake of complement, some results concerning some orders and types of the power sets of polynomials.

Recall the definitions of the order ω and the type τ [4], the logarithmic order ρ^* and logarithmic type Γ^* [8], and the q–order $\rho_{(q)}$ and the q–type $\Gamma_{(q)}$ [7] of the basic set $\{p_n(z)\}$, respectively, in the forms

$$\omega = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\ln \omega_n(r)}{n \ln n}, \tag{4.1}$$

if $0 < \omega < \infty$, then

$$\tau = \lim_{r\to\infty} \frac{e}{\omega} \left[\limsup_{n\to\infty} \{\omega_n(r)\}^{\frac{1}{n}} n^{-\omega} \right]^{\frac{1}{\omega}}, \tag{4.2}$$

$$\rho^* = 1 + \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\ln\{(\omega_n(r))^{\frac{1}{n}}\}}{\ln \ln n}, \tag{4.3}$$

if $1 < \rho^* < \infty$, then

$$\Gamma^* = \frac{1}{\rho^*} \left(\frac{1}{\rho^* - 1} \right)^{\frac{\rho^*-1}{\rho^*}} \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\{\omega_n(r)\}^{\frac{1}{n(\rho^*-1)}}}{\ln \ln n}, \tag{4.4}$$

$$\rho_{(q)} = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\ln \omega_n(r)}{n \ln^{[q-1]} n} \tag{4.5}$$

if $0 < \rho_{(q)} < \infty$, then

$$\Gamma_{(q)} = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\{\omega_n(r)\}^{\frac{1}{n\rho_{(q)}}}}{\ln^{[q-2]} n}, \tag{4.6}$$

where

$$\omega_n(r) = \sum_{k=0}^{\infty} |\pi_{n,k}| M(p_k; r), \tag{4.7}$$

and $M(p_k;r) = \max_{|z|=r} |p_k(z)|$.

We have the following results.

Theorem 4.1. Let $\{p_n(z)\}$ be a simple monic set of polynomials whose coefficients satisfy the condition of the form

$$|p_{n,k}| \leq M n^{\lambda(n-k)}, \quad 0 \leq k \leq n-1. \tag{4.8}$$

Then the power set $\{p_n^{(\mu)}(z)\}$ is of order not exceeding λ.

Theorem 4.2. Let $\{p_n(z)\}$ be a simple monic set of polynomials of order λ and of finite type. Then the power set $\{p_n^{(\mu)}(z)\}$ is of order ρ and does not exceed $(2\mu-1)\lambda$ and is of type zero if the equality sign happens.

Theorem 4.3. Let $\{p_n(z)\}$ be a simple monic set of polynomials of logarithmic order λ^* and of finite logarthmic type. Then the power set $\{p_n^{(\mu)}(z)\}$ is of logarithmic order

$$\rho^* \leq (2\mu-1)\lambda^* - (2\mu-2), \tag{4.9}$$

and is of logarithmic type zero whenever $\rho^* = (2\mu-1)\lambda^* - (2\mu-2)$.

Theorem 4.4. Let $\{p_n(z)\}$ be a simple monic set of polynomials of $q-$order $\rho_{(q)}$ and of finite $q-$type. Then the power set $\{p_n^{(\mu)}(z)\}$ is of $q-$order

$$\rho_{(q)} \leq (2\mu-1)\lambda_{(q)}, \tag{4.10}$$

and is of $q-$type zero whenever $\rho_{(q)} = (2\mu-1)\lambda_{(q)}$.

Theorem 4.5. Let $\{p_n(z)\}$ be a simple set of polynomials of order λ and finite type which satisfies the following condition:

$$\begin{cases} \limsup_{n\to\infty} \frac{\ln|p_{n,n}|}{n\ln n} = b \\ \liminf_{n\to\infty} \frac{\ln|p_{n,n}|}{n\ln n} = a. \end{cases} \tag{4.11}$$

Then the power set $\{p_n^{(\mu)}(z)\}$ is of order

$$\rho \leq (2\mu-1)\lambda + (\mu-1)(b-a). \tag{4.12}$$

If $\rho = (2\mu-1)\lambda + (\mu-1)(b-a)$, then the type of the power set is zero.

Theorem 4.6. Let $\{p_n(z)\}$ be a simple set of polynomials of logarithmic order λ^* and finite logarithmic type which satisfies the following condition:

$$\begin{cases} \limsup_{n\to\infty} \frac{\ln|p_{n,n}|^{\frac{1}{n}}}{\ln\ln n} = b \\ \liminf_{n\to\infty} \frac{\ln|p_{n,n}|^{\frac{1}{n}}}{\ln\ln n} = a. \end{cases} \tag{4.13}$$

Then the power set $\{p_n^{(\mu)}(z)\}$ is of logarithmic order

$$\rho^* \leq (2\mu - 1)\lambda^* + (\mu - 1)(b - a - 2) \tag{4.14}$$

and logarithmic type zero whenever $\rho^* = (2\mu - 1)\lambda^* + (\mu - 1)(b - a - 2)$.

Theorem 4.7. Let $\{p_n(z)\}$ be a simple set of polynomials of q–order $\lambda_{(q)}$ and finite q–type which satisfies the following condition:

$$\begin{cases} \limsup_{n\to\infty} \frac{\ln|p_{n,n}|}{n\ln^{[q-1]} n} = b \\ \liminf_{n\to\infty} \frac{\ln|p_{n,n}|}{n\ln^{[q-1]} n} = a. \end{cases} \tag{4.15}$$

Then the power set $\{p_n^{(\mu)}(z)\}$ is of $q-$order

$$\rho_{(q)} \leq (2\mu - 1)\lambda_{(q)} + (\mu - 1)(b - a). \tag{4.16}$$

These upper bounds in the above theorems are attainable.

Proof of Theorem 4.6. We give as a model the proof of Theorem 4.6 as follows.

Let $\{p_n(z)\}$ be a simple set of polynomials of logarithmic order λ^* and of finite logarithmic type which satisfies (4.13). Suppose that σ is a positive finite number such that $\sigma > \lambda^* - 1$. From (1.6), we have

$$\omega_n(r) < K(\ln n)^{\sigma n}; \quad n \geq 1. \tag{4.17}$$

From (4.13) we get

$$K(\ln n)^{n(a-\epsilon)} < |p_{n,n}| < K(\ln n)^{n(b+\epsilon)}; \quad \epsilon > 0. \tag{4.18}$$

Since the set $\{p_n(z)\}$ is simple, then $p_{n,n}\bar{p}_{n,n} = 1$. Using Cauchy's inequality and from (4.17) and (4.18) we have

$$|p_{n,k}| < K\frac{(\ln n)^{(\sigma+b+\epsilon)n}}{r^k}; \quad k \geq 0, \tag{4.19}$$

and

$$|\bar{p}_{n,k}| < K\frac{(\ln n)^{(\sigma-a+\epsilon)n}}{r^k}; \quad k \geq 0. \tag{4.20}$$

Hence,

$$\begin{aligned} M(p_n^{(\mu)}; r) &= \max_{|z|=r} |p_n^{(\mu)}(z)| \qquad (4.21) \\ &< KM(p_n; r)(\ln n)^{(\mu-1)(\sigma+b+\epsilon)n}\frac{1}{r^{(\mu-1)n}}. \end{aligned}$$

If $\phi_n(r)$ is the Cannon sum of the power set $\{p_n^{(\mu)}(z)\}$, then by using (4.20) and (4.21) we find

$$\begin{aligned} \phi_n(r) &= \sum_k |\bar{p}_{n,k}^{(\mu)}| M(p_k^{(\mu)}; r) \\ &< K(\ln n)^{(2\mu-1)n\sigma+(\mu-1)(b-a)n+2(\mu-1)\epsilon}\frac{1}{r^{(2\mu-3)n}}. \qquad (4.22) \end{aligned}$$

Then the logarithmic order ρ^* of the power set $\{p_n^{(\mu)}(z)\}$ is such that

$$\begin{aligned} \rho^* &= 1 + \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\ln\{(\phi_n(r))^{\frac{1}{n}}\}}{\ln\ln n} \\ &\leq 1 + (2\mu-1)\sigma^* + (\mu-1)(b-a). \end{aligned}$$

Since σ can be taken very close to $\lambda^* - 1$, then

$$\rho^* \leq (2\mu - 1)\lambda^* + (\mu - 1)(b - a - 2). \tag{4.23}$$

Suppose that $\rho^* = (2\mu - 1)\lambda^* + (\mu - 1)(b - a - 2)$. Then for (4.22), we get the logarithmic type as given:

$$\Gamma^* = \frac{1}{\rho^*}\left(\frac{1}{\rho^* - 1}\right)^{\frac{\rho^*-1}{\rho^*}} \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\{\omega_n(r)\}^{\frac{1}{n(\rho^*-1)}}}{\ln\ln n} = 0.$$

The following example shows that the upper bound in (4.23) is attainable.

Example 4.1. Let $\{p_n(z)\}$ be the simple set given by

$$p_n(z) = \begin{cases} b_n z^n + \alpha_n(z^{n-1} + z^{n-2}); & n \quad \text{even} \\ a_n(z^n + z^{n-1}); & n \quad \text{odd} \end{cases}$$

where $a_n = (\ln n)^{an+2}$, $\quad b_n = (\ln n)^{bn+3}$, and $\alpha_n = (\ln n)^{(\alpha+b-1)n}$. Hence, $\lambda^* = \alpha$.

For $\mu = 2$, we construct the power set $\{p_n^{(2)}(z)\}$ as follows:

$$\begin{aligned} p_n^{(2)}(z) &= b_n^2 z^n + \alpha_n(b_n + a_{n-1})z^{n-1} \\ &\quad + \alpha_n(b_n + b_{n-2} + a_{n-1})z^{n-2} + \alpha_n\alpha_{n-1}(z^{n-3} + z^{n-4}); \quad n \text{ even} \end{aligned}$$

and

$$p_n^2(z) = a_n^2 z^n + a_n(a_n + b_{n-1})z^{n-1} + a_n\alpha_{n-1}(z^{n-2} + z^{n-3}); \quad n \text{ odd.}$$

Then the logarithmic order ρ^* of the power set $\{p_n^{(2)}(z)\}$ is such that

$$\rho^* = 1 + \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\ln\{(\phi_n(r))^{\frac{1}{n}}\}}{\ln\ln n} = 3\alpha + b - a - 2$$

where a, b and α are real positive numbers. Hence, the proof of Theorem 4.6 is completed.

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