SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

145. [2004, 58] *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Let F_n denote the nth Fibonacci number ($F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$) and let L_n denote the nth Lucas number ($L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$). Prove that

$$F_{n+1} > \frac{1}{3}\left(\frac{L_n^{L_n}}{F_n^{F_n}}\right)^{\frac{1}{L_n - F_n}}$$

holds for all positive integer $n \geq 2$.

Solution by the proposer. It is well known [1] that for a positive integrable function defined on the interval $[a, b]$, the integral analogue of the AM-GM inequality is given by

$$A(f) = \frac{1}{b-a}\int_a^b f(x)\,dx \geq \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right) = G(f). \tag{1}$$

Setting $f(x) = x$, $a = F_n$, and $b = L_n$ into (1), yields

$$\frac{1}{L_n - F_n}\int_{F_n}^{L_n} x\,dx \geq \exp\left(\frac{1}{L_n - F_n}\int_{F_n}^{L_n} \ln x\,dx\right).$$

Note that for all $n \geq 2$, $L_n - F_n > 0$. Evaluating the preceding integrals and after simplification, we obtain

$$\begin{aligned}\frac{F_n + L_n}{2} &\geq \exp\left(\frac{1}{L_n - F_n}\ln\left(\frac{L_n^{L_n}}{F_n^{F_n}}\right) - 1\right) \\ &= \exp\left(\ln\left[\frac{1}{e}\left(\frac{L_n^{L_n}}{F_n^{F_n}}\right)\right]^{\frac{1}{L_n - F_n}}\right). \end{aligned} \tag{2}$$

Since $F_n + L_n = 2F_{n+1}$, as can be easily proved by mathematical induction, then (2) becomes

$$F_{n+1} \geq \frac{1}{e}\left(\frac{L_n^{L_n}}{F_n^{F_n}}\right)^{\frac{1}{L_n - F_n}} > \frac{1}{3}\left(\frac{L_n^{L_n}}{F_n^{F_n}}\right)^{\frac{1}{L_n - F_n}}$$

and we are done.

References

1. G. H. Hardy, J. E. Littlewood, & G. Pólya, *Inequalities,* Cambridge University, Press, UK, 1997, pp. 137–138.

Also solved by Said Amghibech, Sainte Foy (qc), Canada. One partial solution was also received.

146. [2004, 58] *Proposed by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri.*

Find all (x, y) with $0 \leq x < 2\pi$ and $0 \leq y < 2\pi$ such that

$$\cos^2 y = 2(\sin x + \cos x \cos y - 1).$$

Solution by Mihai Cipu, Romanian Academy, Bucharest, Romania; Said Amghibech, Sainte Foy (qc), Canada; Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan; and the proposer (independently). All the solutions were essentially the same. The given equation can be written as

$$(\sin x - 1)^2 + (\cos x - \cos y)^2 = 0.$$

Since the sum of two nonnegative real numbers can be 0 if and only if each of the terms is 0,

$$\sin x = 1 \quad \text{and} \quad \cos x = \cos y.$$

That is,

$$\sin x = 1 \quad \text{and} \quad \cos y = 0.$$

Therefore,

$$(x, y) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \text{or} \quad (x, y) = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

147. [2004, 59] *Proposed by Zdravko F. Starc, Vršac, Serbia and Montenegro.*

Let F_n be the Fibonacci numbers defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that

$$(F_1^4 + F_{n+1}^4)(F_2^4 + F_{n+2}^4) \cdots (F_n^4 + F_{2n}^4) < \left(\frac{2^{4n-1}}{n} \right)^{2n}.$$

Solution I by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan. The following inequality is valid:

$$a^4 + b^4 \leq (a^2 + b^2)^2 \quad \text{for all} \quad a, b \in \mathbb{R}.$$

Thus,

$$(F_1^4 + F_{n+1}^4)(F_2^4 + F_{n+2}^4) \cdots (F_n^4 + F_{2n}^4) < \Big((F_1^2 + F_{n+1}^2) \cdots (F_n^2 + F_{2n}^2) \Big)^2.$$

On the other hand we get by the AM-GM inequality that

$$(F_1^2 + F_{n+1}^2) \cdots (F_n^2 + F_{2n}^2) \leq \left(\frac{F_1^2 + F_2^2 + \cdots + F_{2n}^2}{n} \right)^n.$$

Therefore, we obtain that

$$(F_1^4 + F_{n+1}^4) \cdots (F_n^4 + F_{2n}^4) \leq \left(\frac{F_1^2 + F_2^2 + \cdots + F_{2n}^2}{n} \right)^{2n}.$$

It suffices to show that

$$F_1^2 + F_2^2 + \cdots + F_{2n}^2 \leq 2^{4n-1}.$$

But observe that

$$F_1^2 + \cdots + F_{2n}^2 = F_{2n} F_{2n+1}.$$

Also by Binet's formula

$$F_{2n}F_{2n+1} = \frac{1}{5}(\alpha^{2n} - \beta^{2n})(\alpha^{2n+1} - \beta^{2n+1})$$

$$= \frac{1}{5}(\alpha^{4n+1} - (\alpha\beta)^{2n} \cdot \beta - (\beta\alpha)^{2n} \cdot \alpha + \beta^{4n+1}).$$

Here,

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Since

$$\alpha\beta = -1, \quad \alpha + \beta = 1, \quad \beta^{4n+1} = \beta^{4n} \cdot \beta < 0, \quad \text{and} \quad \alpha = \frac{1+\sqrt{5}}{2} < 2,$$

we have that

$$F_{2n}F_{2n+1} = \frac{1}{5}(\alpha^{4n+1} - \alpha - \beta + \beta^{4n+1})$$

$$< \frac{1}{5} \cdot \alpha^{4n+1} < \frac{2^{4n+1}}{5} = \frac{4}{5} \cdot 2^{4n-1} < 2^{4n-1}.$$

The result follows.

Solution II by Mihai Cipu, Romanian Academy, Bucharest, Romania. We shall prove a stronger inequality: the product on the left hand side is less than $2^{(9n^2+n)/2}$.

Let us denote $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$. Since by Binet's formula $F_n = (a^n = b^n)/(a-b)$ for any n, it is easy to prove by induction that $F_n < 2a^{n-2}$ for $n \geq 2$. Therefore, for $n \geq 1$ we have

$$\prod_{i=1}^{n} F_{n+i} < \prod_{i=1}^{n} 2a^{n+i-2} = 2^n a^{3(n^2-n)/2}.$$

Hence,

$$\prod_{i=1}^{n}(F_i^4 + F_{n+i}^4) < \prod_{i=1}^{n} 2F_{n+i}^4 < 2^{5n}a^{6(n^2-n)}.$$

¿From $a < 5/3 < 2^{3/4}$ it follows that the given product is less than

$$2^{5n+9(n^2-n)/2} = 2^{(9n^2+n)/2}.$$

To show that this upper bound is better than that given in the problem, we have to prove

$$n^2 2^{(9n+1)/2} < 2^{8n-2},$$

or $n^4 < 2^{7n-5}$ for $n \geq 2$, which is readily obtained by induction, since $128n^4 > (n+1)^4$.

Also solved by Said Amghibech, Sainte Foy (qc), Canada and the proposer.

148. [2004, 59] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Show that

$$\prod_{i=1}^{\infty}\left(\frac{\cos\frac{x}{4^i} + \cos\frac{3x}{4^i}}{2}\right) = \prod_{i=1}^{\infty}\left(\frac{1 + 2\cos\frac{2x}{5^i} + 2\cos\frac{4x}{5^i}}{5}\right),$$

where x is any real or complex number.

Solution I by Mihai Cipu, Romanian Academy, Bucharest, Romania. Both sides of the proposed equality are 1 if $x = 0$, so in the following we shall assume $x \neq 0$.

As

$$\cos\frac{x}{4^i} + \cos\frac{3x}{4^i} = 2\cos\frac{x}{4^i}\cos\frac{2x}{4^i}$$

and

$$\prod_{i=1}^{n}\cos 2^{i-1}x = \frac{\sin 2^n x}{2^n \sin x},$$

we have

$$P_n := \prod_{i=1}^{n} \left(\frac{\cos \frac{x}{4^i} + \cos \frac{3x}{4^i}}{2} \right) = \prod_{i=1}^{n} \cos \frac{x}{2^{2i}} \cos \frac{x}{2^{2i-1}} = \frac{\sin x}{4^n \sin \frac{x}{4^n}}.$$

Hence, the product on the left hand side of the proposed equality is

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{\sin x}{4^n \sin \frac{x}{4^n}} = \frac{\sin x}{x} \lim_{n\to\infty} \frac{\frac{x}{4^n}}{\sin \frac{x}{4^n}} = \frac{\sin x}{x}.$$

In the right hand side we shall use the identity

$$1 + 2\cos 2x + 2\cos 4x = \frac{\sin x + \sin 3x - \sin x + \sin 5x - \sin 3x}{\sin x} = \frac{\sin 5x}{\sin x}.$$

Therefore,

$$Q_n := \prod_{i=1}^{n} \frac{\sin \frac{x}{5^{i-1}}}{5 \sin \frac{x}{5^i}} = \frac{\sin x}{5^n \sin \frac{x}{5^n}}$$

and

$$\prod_{i=1}^{n} \left(\frac{1 + 2\cos \frac{2x}{5^i} + 2\cos \frac{4x}{5^i}}{5} \right) = \lim_{n\to\infty} Q_n = \lim_{n\to\infty} \frac{\sin x}{5^n \sin \frac{x}{5^n}} = \frac{\sin x}{x}.$$

Solution II by Larry Eifler, University of Missouri - Kansas City, Kansas City, Missouri. We establish a more general result since this will better illustrate the patterns underlying the result. If A and B are complex numbers, then

$$2\sin A\cos B = \sin(A+B) - \sin(B-A).$$

Let m be a positive integer. Using the above identity, we see that

$$\begin{aligned}\sin\theta\left(\frac{\sum_{k=1}^{m}\cos(2k-1)\theta}{m}\right) &= \frac{\sum_{k=1}^{m}[\sin 2k\theta - \sin(2k-2)\theta]}{2m}\\ &= \frac{\sin 2m\theta}{2m}\end{aligned}$$

and

$$\begin{aligned}\sin\theta\left(\frac{1+2\sum_{k=1}^{m}\cos 2k\theta}{2m+1}\right) &= \frac{\sin\theta + \sum_{k=1}^{m}[\sin(2k+1)\theta - \sin(2k-1)\theta]}{2m+1}\\ &= \frac{\sin(2m+1)\theta}{2m+1}.\end{aligned}$$

Let x be a complex number. If $\sin x \neq 0$, then

$$\begin{aligned}\prod_{i=1}^{n}\left(\frac{\sum_{k=1}^{m}\cos\frac{(2k-1)x}{(2m)^i}}{m}\right) &= \prod_{i=1}^{n}\left(\frac{\sin\frac{2mx}{(2m)^i}}{2m\sin\frac{x}{(2m)^i}}\right)\\ &= \frac{\sin x}{(2m)^n\sin\frac{x}{(2m)^n}}\end{aligned}$$

and

$$\begin{aligned}\prod_{i=1}^{n}\left(\frac{1+2\sum_{k=1}^{m}\cos\frac{2kx}{(2m+1)^i}}{2m+1}\right) &= \prod_{i=1}^{n}\left(\frac{\sin\frac{(2m+1)x}{(2m+1)^i}}{(2m+1)\sin\frac{x}{(2m+1)^i}}\right)\\ &= \frac{\sin x}{(2m+1)^n\sin\frac{x}{(2m+1)^n}}.\end{aligned}$$

Thus,

$$\prod_{i=1}^{n}\left(\frac{\sum_{k=1}^{m}\cos\frac{(2k-1)x}{(2m)^i}}{m}\right)=\frac{\sin x}{x}\frac{\frac{x}{(2m)^n}}{\sin\frac{x}{(2m)^n}}\quad\text{if}\quad\sin\frac{x}{(2m)^n}\neq 0$$

since the functions in the above formula are continuous at x if $\sin\frac{x}{(2m)^n}\neq 0$.

Similarly,

$$\prod_{i=1}^{n}\left(\frac{1+2\sum_{k=1}^{m}\cos\frac{2kx}{(2m+1)^i}}{2m+1}\right)=\frac{\sin x}{x}\frac{\frac{x}{(2m+1)^n}}{\sin\frac{x}{(2m+1)^n}}\quad\text{if}\quad\sin\frac{x}{(2m+1)^n}\neq 0.$$

Hence,

$$\prod_{i=1}^{\infty}\left(\frac{\sum_{k=1}^{m}\cos\frac{(2k-1)x}{(2m)^i}}{m}\right)=\frac{\sin x}{x}=\prod_{i=1}^{\infty}\left(\frac{1+2\sum_{k=1}^{m}\cos\frac{2kx}{(2m+1)^i}}{2m+1}\right)\quad\text{for}\quad x\neq 0$$

since $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$. The two infinite products in the above formula are clearly equal at $x=0$.

Also solved by Said Amghibech, Sainte Foy (qc), Canada and the proposer. A partial solution was also received.