ON A TAXICAB DISTANCE ON A SPHERE

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Abstract. In this work, we define a *spherical taxicab distance* on the surface of a sphere in terms of well-known geographical latitude and longitude. Then we study some properties of this distance function and give an analogue of the ruler postulate. At the end, we give some connections between the spherical taxicab circle and the spherical circles.

1. Introduction. At the beginning of the last century, a family of metrics for plane geometries, including the taxicab metric, was published in [8]. Later, taxicab plane geometry was introduced in [7] and developed in [5] using the taxicab distance

$$|x_1 - x_2| + |y_1 - y_2|$$

instead of the Euclidean distance

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

between the points (x_1, y_1) and (x_2, y_2) in the coordinate plane.

A few problems related to the taxicab geometry have been studied and improved by some authors [1, 2, 3, 6, 9, 10, 11, 12, 13, 15]. It is stated in [5] that while the Euclidean geometry appears to be a good model of the natural world, the taxicab geometry is a better model of the artificial urban world that man has built. Since the surface of the world resembles the surface of a sphere more than the plane, it comes to mind that the spherical taxicab geometry may be more meaningful than plane geometry. Therefore, in this work, we give a taxicab distance on a surface of a sphere and introduce spherical taxicab geometry.

2. The Spherical Distance Between Two Points On the Sphere. Let u and v be *latitude* and *longitude* of a given point P on the sphere, respectively. The ordered pair (u, v) is called the *geographical coordinates* of this point, where $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$ and $-\pi \le v \le \pi$. P is on the northern hemisphere if and only if $0 \le u \le \frac{\pi}{2}$ and P is on the southern hemisphere if and only if $-\frac{\pi}{2} \le u < 0$.

Similarly, P is on the eastern hemisphere if and only if $0 \leq v \leq \pi$ and P is on the western hemisphere if and only if $-\pi < v < 0$.

Thus, the North Pole is $N = (-\frac{\pi}{2}, 0)$. Now let $P = (u_1, v_1)$ and $Q = (u_2, v_2)$ be any two distinct points on the sphere. Consider the spherical triangle NPQ. (The closed geometric figure is formed on the surface of the sphere and is bounded by intersecting minor arcs of three great circles as in Figure 1). The sides of the spherical triangle NPQ on the unit sphere are the following circular arcs:

$$a = \angle POQ := \overset{\frown}{PQ},\ b = \angle QON := \overset{\frown}{QN} \text{ and } c = \angle PON := \overset{\frown}{PN}.$$

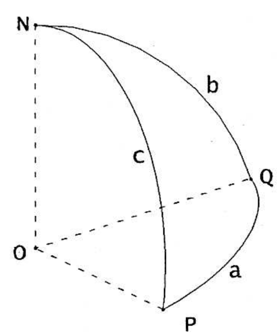

Figure 1.

The angles of the triangles NPQ are the spherical angles formed at the intersection points N, P, and Q of two great circles of the sphere, equal to the angles between their tangents at the point of intersection. It is known [14] that the following connections between a, b, c and the spherical angles $\widehat{N}$, $\widehat{P}$, $\widehat{Q}$ of the spherical triangle NPQ are valid:

$$\begin{aligned} \cos a &= \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos N \\ \cos b &= \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos P \\ \cos c &= \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos Q \end{aligned} \tag{1}$$

Now, the arc length from P to Q is calculated from the first equation in (1) as follows:

$$\cos \overset{\frown}{PQ} = \cos \overset{\frown}{NP} \cdot \cos \overset{\frown}{NQ} + \sin \overset{\frown}{NP} \cdot \sin \overset{\frown}{NQ} \cdot \cos \widehat{N},$$

where

$$\widehat{N} = \begin{cases} |v_1 - v_2|, & \text{if } |v_1 - v_2| \leq \pi \\ 2\pi - |v_1 - v_2|, & \text{if } |v_1 - v_2| > \pi. \end{cases}$$

$$\cos \overset{\frown}{PQ} = \cos\left(\frac{\pi}{2} - u_1\right) \cdot \cos\left(\frac{\pi}{2} - u_2\right) + \sin\left(\frac{\pi}{2} - u_1\right) \cdot \sin\left(\frac{\pi}{2} - u_2\right) \cdot \cos(v_1 - v_2)$$

and

$$\overset{\frown}{PQ} = \arccos(\sin u_1 \cdot \sin u_2 + \cos u_1 \cdot \cos u_2 \cdot \cos(v_1 - v_2)). \tag{2}$$

Thus, for the *spherical arc length,* $d_S(P,Q)$, that is the *spherical distance* from P to Q on the sphere with radius r, one has

$$d_S(P,Q) = \left|\overset{\frown}{PQ}\right| = r \cdot \arccos\left(\sin u_1 \cdot \sin u_2 + \cos u_1 \cdot \cos u_2 \cdot \cos\left[|v_1 - v_2|\right]\right), \tag{3}$$

where

$$[|v_1 - v_2|] := \begin{cases} |v_1 - v_2|, & \text{if } |v_1 - v_2| \leq \pi \\ 2\pi - |v_1 - v_2|, & \text{if } |v_1 - v_2| > \pi \ . \end{cases}$$

3. The Taxicab Distance Between Two Points on Sphere.

Definition 1. Let $P = (u_1, v_1)$ and $Q = (u_2, v_2)$ be any two points with the geographical coordinates on a sphere of radius r. Let C be the intersection point of the parallel of latitude on P and the meridian on Q, and D be the intersection point of the parallel of the latitude on Q and the meridian on P. (See Figure 2). Then,

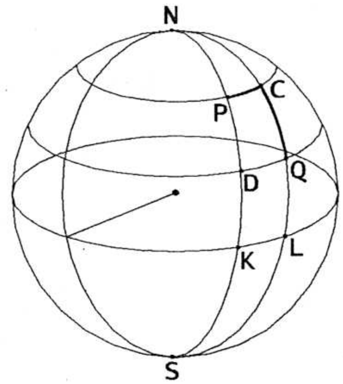

Figure 2.

$$d_{TS}(P,Q) := \min\left\{\left|\widehat{PC}\right| + \left|\widehat{CQ}\right|, \left|\widehat{PD}\right| + \left|\widehat{DQ}\right|\right\} \tag{4}$$

is called The Spherical Taxicab Distance between P and Q.

Since

$$\left|\widehat{PD}\right| = \left|\widehat{CQ}\right|,$$

$$d_{TS}(P,Q) = \begin{cases} \left|\widehat{PC}\right| + \left|\widehat{CQ}\right|, & |u_1| \geq |u_2| \\ \left|\widehat{PD}\right| + \left|\widehat{DQ}\right|, & |u_1| < |u_2| . \end{cases} \tag{5}$$

The latitude of the point C is u_1 since C is on the same parallel with P, and the longitude of point C is v_2 since C is on the same meridian with Q. Similarly, the latitude of D is u_2 and the longitude of D is v_1.

The arc lengths $\widehat{PC}$ and $\widehat{DQ}$, in terms of the length of $\widehat{KL}$ on the equator are

$$\left|\widehat{PC}\right| = \left|\widehat{KL}\right| \cdot \cos u_1 = \begin{cases} r\,|v_1 - v_2| \cos u_1, & \text{if } |v_1 - v_2| \leq \pi \\ r(2\pi - |v_1 - v_2|) \cos u_1, & \text{if } |v_1 - v_2| > \pi \end{cases}$$

and

$$\left|\widehat{DQ}\right| = \left|\widehat{KL}\right| \cdot \cos u_2 = \begin{cases} r\,|v_1 - v_2| \cos u_2, & \text{if } |v_1 - v_2| \leq \pi \\ r(2\pi - |v_1 - v_2|) \cos u_2, & \text{if } |v_1 - v_2| > \pi \ . \end{cases}$$

Also,

$$\left|\widehat{CQ}\right| = \left|\widehat{PD}\right| = r \cdot |u_1 - u_2| \, .$$

Now the spherical taxicab distance between P and Q can be formulated as follows:

$$d_{ST}(P,Q) = \begin{cases} r\left(|u_1 - u_2| + |v_1 - v_2| \cos u_1\right), & |v_1 - v_2| \leq \pi \text{ and } |u_1| \geq |u_2| \\ r\left(|u_1 - u_2| + (2\pi - |v_1 - v_2|) \cos u_1\right), & |v_1 - v_2| > \pi \text{ and } |u_1| \geq |u_2| \\ r\left(|u_1 - u_2| + |v_1 - v_2| \cos u_2\right), & |v_1 - v_2| \leq \pi \text{ and } |u_1| < |u_2| \\ r\left(|u_1 - u_2| + (2\pi - |v_1 - v_2|) \cos u_2\right), & |v_1 - v_2| > \pi \text{ and } |u_1| < |u_2| \ . \end{cases} \tag{6}$$

The last formula can also be shortened as

$$d_{ST}(P,Q) = r.\left\{|u_1 - u_2| + [|v_1 - v_2|] \cos u_i\right\}, \quad u_i = \max\left\{|u_1|, |u_2|\right\}, \tag{7}$$

where

$$[|v_1 - v_2|] := \begin{cases} |v_1 - v_2| & \text{if } |v_1 - v_2| \leq \pi \ , \\ 2\pi - |v_1 - v_2| & \text{if } |v_1 - v_2| > \pi \quad . \end{cases}$$

The following proposition asserts that the taxicab distance function is *positive definite* and *symmetric* but doesn't satisfy the *triangle inequality:*

Proposition 1.

a) To each ordered pair of points (P,Q) on a sphere, d_{ST} assigns a non-negative number $d_{ST}(P,Q)$. Furthermore, $d_{ST}(P,Q) = 0$ if and only if $P = Q$.
b) $d_{ST}(P,Q) = d_{ST}(Q,P)$,
c) The inequality $d_{ST}(P,Q) + d_{ST}(Q,R) \geq d_{ST}(P,R)$ is not valid.

Proof. The properties a) and b) can be easily seen from the definition of d_{ST}. In order to see that d_{ST} doesn't satisfy the triangle inequality, consider the triangle PQR with $P = (\pi/12, 0), Q = (5\pi/12, \pi/6)$ and $R = (0, \pi)$.

Now, let's answer the following question: *How can we give an analogue of the ruler postulate for the spherical taxicab geometry?*

Let $\mathcal{P}$ denote the set of all points on the sphere of radius r and $P, Q \in \mathcal{P}$. We define PQ-path as union of the arcs used to calculate the spherical taxicab distance from P to Q. Let $\mathcal{L}$ denote the set of all XY-paths on the sphere,

$$\mathcal{L} := \{XY\text{-path } : X \text{ and } Y \text{ any two points on the sphere}\}.$$

Thus, the set $\mathcal{P}$ of points and the set $\mathcal{L}$ of XY-paths on the sphere form a geometrical structure $(\mathcal{P}, \mathcal{L})$. One can use the function

$$f(u) = (2u + \pi \cos u)r$$

to calculate the length of the longest path (that is, the maximum spherical distance between any two points P and Q on the sphere). Here, u is the latitude of P or Q, which is nearer to a pole. For the maximum of f,

$$\begin{aligned} u &\cong 39,540223747810195412699015559026 1^\circ \\ &\cong 39^\circ 32' 24'' \\ &\cong 0,219667909710056641181661197550145\pi \\ &\cong 0,690107091374539952004377909070395 \end{aligned}$$

and the maximum value k of f,

$$\begin{aligned} k &\cong 1,21051366235301868432776943516072\pi r \\ &\cong 3,80294082871831896417830842653785 r. \end{aligned}$$

Hence,

$$0 \le f(u) \le k \Rightarrow \max d_{ST}(P, Q) \le k.$$

Let PQ-path be a path on the sphere which is union of the arcs $\overset{\frown}{PC}$ and $\overset{\frown}{CQ}$, where $P = (u_0, v_0)$, $Q = (u_0', v_0')$ and $C = (u_0, v_0')$. That is, let the point P be the nearer point to a pole. Now we define the function φ as follows:

$$\varphi\colon PQ - \text{path} \longrightarrow [0, d_{ST}(P,Q)]$$

$$(u,v) \longrightarrow \varphi(u,v) = \begin{cases} [|v - v_0|] \cos u_0, & \text{if } (u,v) \text{ is on } \overset{\frown}{PC} \\ |u - u_0| + [|v - v_0|] \cos u_0, & \text{if } (u,v) \text{ is on } \overset{\frown}{CQ} \end{cases},$$

where

$$[|v - v_0|] := \begin{cases} |v - v_0|, & \text{if } |v - v_0| \leq \pi \\ 2\pi - |v - v_0|, & \text{if } |v - v_0| > \pi \end{cases}.$$

The function φ can be considered as a ruler for PQ-path since φ is a bijection and satisfies the Ruler Equation

$$|\varphi(X) - \varphi(Y)| = d_{ST}(X,Y),$$

where X and Y is any two points on PQ-path.

4. Spherical Circle and Spherical Taxicab Circle.

a. *Equation of a spherical circle with a center on the sphere in terms of geographical coordinates.*

Clearly, intersection of a plane and a sphere is a circle if they meet. A *spherical circle* is defined as the locus of points that are equispherical distant from a given fixed point on a sphere. Now, consider a sphere with radius r and let $P = (u_0, v_0)$ be a point on this sphere. Let's find the equation of the spherical circle consisting of points with k units distance from P measured along the great circles passing through P. If $X = (u,v)$ is on this spherical circle, then $|\overset{\frown}{PX}| = k$ (see Figure.3). Now, applying the first equality in (1), one gets

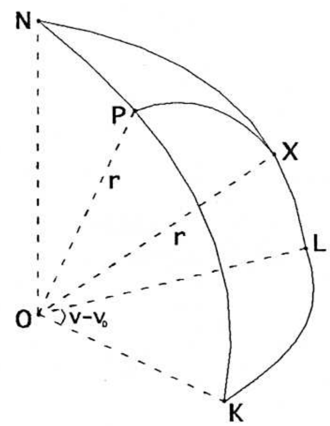

Figure 3.

$$\cos \widehat{PX} = \cos \widehat{NP} \cdot \cos \widehat{NX} + \sin \widehat{NP} \cdot \sin \widehat{NX} \cdot \cos \widehat{N}$$

$$\cos \frac{k}{r} = \cos\left(\frac{\pi}{2} - u_0\right) \cdot \cos\left(\frac{\pi}{2} - u\right) + \sin\left(\frac{\pi}{2} - u_0\right) \cdot \sin\left(\frac{\pi}{2} - u\right) \cdot \cos(v - v_0)$$

$$k = r \arccos(\sin u_0 \sin u + \cos u_0 \cos u \cos(v - v_0)), \quad k \leq \pi r. \tag{8}$$

b. *The Spherical Taxicab Circle.*

We define the *spherical taxicab circle* as the set of points that have a constant spherical taxicab distance from a given fixed point on a sphere. Let's find the equation of the spherical taxicab circle with radius k, $k \leq \pi r$, $k \in R$ and with center $P(u_0, v_0)$ on a sphere with radius r. If $X = (u, v)$ is on the spherical taxicab circle, then

$$d_{ST}(P, Q) = k, \quad k \leq \pi r,$$

which is equivalent to

$$k = \begin{cases} r\left(|v-v_0|\cos u + |u-u_0|\right), & |v-v_0| \leq \pi \text{ and } |u| \geq |u_0| \\ r\left((2\pi - |v-v_0|)\cos u + |u-u_0|\right), & |v-v_0| > \pi \text{ and } |u| \geq |u_0| \\ r\left(|v-v_0|\cos u_0 + |u-u_0|\right), & |v-v_0| \leq \pi \text{ and } |u| < |u_0| \\ r\left((2\pi - |v-v_0|)\cos u_0 + |u-u_0|\right), & |v-v_0| > \pi \text{ and } |u| < |u_0| \,. \end{cases} \tag{9}$$

Proposition 2. A spherical circle and a spherical taxicab circle with the same radius k coincide if they have a common center at a pole.

Proof. Let the North Pole $(u_0, v_0) = (\frac{\pi}{2}, 0)$ be the center of both circles. Then, using equation (8), we get

$$k = r \arccos{(\sin u)}, \qquad k \leq \pi r$$

as the equation of the spherical circle, which means

$$k = r\left(\frac{\pi}{2} - u\right), \qquad k \leq \pi r.$$

For the equation of the spherical taxicab circle with radius k, using Equation (9), we get

$$k = r\left(|v|\cos\frac{\pi}{2} + \left|u - \frac{\pi}{2}\right|\right), \qquad |v| \leq \pi, \quad |u| < \frac{\pi}{2}.$$

That is,

$$k = r\left|u - \frac{\pi}{2}\right|, \qquad |v| \leq \pi, \quad |u| < \frac{\pi}{2}$$

$$k = r\left(\frac{\pi}{2} - u\right), \qquad k \leq \pi r.$$

Similarly, if $(u_0, v_0) = (-\frac{\pi}{2}, 0)$, then in both cases one obtains the equation

$$k = r\left(\frac{\pi}{2} + u\right), \qquad k \leq \pi r.$$

In general, the taxicab spherical circles are not Euclidean circles, except with the center at a pole. It also seems that they are not a combination of circular arcs, and are not planar curves.

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