NOTES ON THE STRUCTURE OF $\mathrm{P}\Sigma_{\mathbf{n}}$

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Abstract. The pure symmetric automorphism group, $\mathrm{P}\Sigma_{\mathbf{n}}$, consists of those automorphisms of the free group on n generators that take each standard generator to a conjugate of itself. We give presentations for kernels of homomorphisms $\mathrm{P}\Sigma_{\mathbf{n}} \to \mathbb{Z}$ where each standard generator is sent to either 0 or 1, and provide explicit generators (as words in the standard generators) when those kernels are finitely generated. In addition, we provide recursive constructions of the defining graphs of the graph groups associated with $\mathrm{P}\Sigma_{\mathbf{n}}$.

1. Introduction. The *pure symmetric automorphism group,* $\mathrm{P}\Sigma_{\mathbf{n}}$, consists of those automorphisms of the free group on n generators, F_n, that take each standard generator to a conjugate of itself. Such automorphisms have also been referred to as *basis-conjugating automorphisms of free groups.*

Just as each braid group, B_n, can be interpreted as a group of motions of n distinct points in the plane, so $\mathrm{P}\Sigma_{\mathbf{n}}$ can be interpreted as a group of motions of n unknotted, unlinked circles in $\mathbb{R}^3$. The fundamental group of the link complement of this trivial link is F_n, with each of the n standard generators corresponding to a meridian of one of the circles. $\mathrm{P}\Sigma_{\mathbf{n}}$ is then realized as the group of motions generated by those motions for which the ith circle is passed through the jth circle, with all others remaining fixed. See Goldsmith [7] or Brownstein and Lee [4] for details.

McCool [13], using generators described by Humphries [9], derived a finite presentation for $\mathrm{P}\Sigma_{\mathbf{n}}$. Gutiérrez and Krstić [8] have discovered normal forms for the more general *symmetric automorphism group of* F_n (those that send each standard generator to a conjugate of itself or another standard generator) that form a regular language. Orlandi-Korner [16] has computed the first Σ-invariant for $\mathrm{P}\Sigma_{\mathbf{n}}$. Brady et al. [3] have shown that the $\mathrm{P}\Sigma_{\mathbf{n}}$ are duality groups.

Since two of the three types of relators in McCool's presentation for $\mathrm{P}\Sigma_{\mathbf{n}}$ are commutators of his generators, there is an associated *graph group* or *right-angled Artin group* associated with each $\mathrm{P}\Sigma_{\mathbf{n}}$; $\mathrm{P}\Sigma_{\mathbf{n}}$ is the quotient of this graph group by the normal subgroup generated by the other relators. Graph groups have been studied extensively, and have proven to be an interesting yet accessible class of groups. See [2, 5, 6, 10, 14, 15, and 17] for some results about graph groups.

The focus of this paper is threefold. First, we derive a presentation for kernels of homomorphisms $\theta\colon \mathrm{P}\Sigma_{\mathbf{n}} \to \langle t\rangle$ (where $\langle t\rangle$ is the infinite cyclic group) in which each generator is mapped to t or 1. (Kernels from graph groups to $< t >$ provided the first example of groups that satisfy homological finiteness condition FP_2 but which were not finitely presented [2].) These kernels necessarily contain the commutator subgroup of $\mathrm{P}\Sigma_{\mathbf{n}}$. Secondly, we study the structure of the underlying graph of the graph group associated with $\mathrm{P}\Sigma_{\mathbf{n}}$. Lastly, we provide and explicit finite generating set for those homomorphisms θ for which $\ker\theta$ is finitely generated.

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2. Preliminaries. Let F_n be the free group with basis $X = \{x_1, x_2, \ldots, x_n\}$. The *pure symmetric automorphism group* of F_n, denoted by $\mathrm{P}\Sigma_{\mathbf{n}}$, is the subgroup of $\mathrm{Aut}(F_n)$ that sends each x_i to a conjugate of itself.

In [13], McCool provides a finite presentation of $\mathrm{P}\Sigma_{\mathbf{n}}$. The generators are the maps $\phi_{i,j}\colon F_n \to F_n$ $(i \neq j)$ given by

$$\phi_{i,j}(x_k) = \begin{cases} x_j^{-1} x_i x_j & \text{if } k = i \\ x_k & \text{otherwise} \end{cases}$$

and the relations fall into three categories:

$$\begin{cases} [\phi_{i,j}, \phi_{k,j}] = 1 & \text{for all distinct } i, j \text{ and } k \\ [\phi_{i,j}, \phi_{k,l}] = 1 & \text{for all distinct } i, j, k \text{ and } l \\ [\phi_{i,j}, \phi_{i,k}\phi_{j,k}] = 1 & \text{for all distinct } i, j \text{ and } k. \end{cases} \tag{1}$$

We call the first class of relators Type C; these are relations in which automorphisms with **common** conjugates commute. The second class is Type D; these are relations in which automorphisms with **disjoint** subscripts commute. The third class is Type E; these can be thought of as **edge** relations, where a single vertex commutes with a pair of common conjugates (as defined by the equation above) that correspond to an edge in the underlying graph group defined below.

Since the relators of Types C and D are commutators of generators, the group formed from the presentation above minus the relators of Type E is a *graph group* or *right-angled Artin group.*

Following [12], we will denote the Tietze transformations adjunction of relators, deletion of relators, adjunction of generators, and deletion of generators by (T1), (T2), (T3), and (T4), respectively. It is shown in [12] that given any two presentations of a group G, one can be obtained from the other by repeated applications of these four transformations.

Given a presentation of a group G and suitable information about a subgroup $H \leq G$, the Reidemeister-Schreier method enables one to obtain a presentation for H. In [11] it is shown that if $\langle X; R\rangle$ is a presentation for G, $\pi\colon F(X) \to G$ is the canonical epimorphism, T is a Schreier transversal for $\pi^{-1}(H)$ in $F(X)$, and $\Phi\colon F(X) \to T$ is the function which maps each element to its coset representative, then $Y = \{tx\Phi(tx)^{-1} \mid t \in T,\ x \in X,\ tx \notin T\}$ corresponds to a set of generators for H. Furthermore, if $\tau\colon \pi^{-1}(H) \to F(Y)$ is the function in [11] which rewrites each $w \in \pi^{-1}(H)$ in terms of the generators Y and $S = \{\tau(trt^{-1}) \mid t \in T, r \in R\}$, then $\langle Y; S\rangle$ is a presentation for H.

The reader is directed to [11] or [12] for details of Tietze transformations and the Reidemeister-Schreier rewriting procedure.

3. A Presentation for the Subgroups. Let $\mathrm{P}\Sigma_{\mathbf{n}}$ have the presentation given in Section 2, with generating set $X = \{\phi_{i,j} \mid 1 \leq i, j \leq n,\ i \neq j\}$ and relators given in Equation 1 of Types C, D, and E. Let $\theta\colon \mathrm{P}\Sigma_{\mathbf{n}} \to \langle t\rangle$ be a homomorphism that maps each generator $\phi_{i,j}$ to either t or 1. We seek a presentation for $\ker\theta$.

Following [1] and [10], we will refer to those generators mapped to t as "live" and those mapped to 1 as "dead."

Since there will always be at least one live generator, and the presentation of $\mathrm{P}\Sigma_{\mathbf{n}}$ is symmetric, we will always assume$\phi_{1,2}$ is live.

For a Schreier transversal for $\pi^{-1}(\ker\theta)$ in $F(X)$, take $T = \{\phi_{1,2}^m \mid m \in \mathbb{Z}\}$ and recall that $\Phi\colon F(X) \to T$ sends each element to its coset representative. A set of generators for $\ker\theta$ is then $\{tx\Phi(tx)^{-1} \mid t \in T, x \in X, tx \notin T\}$. Just as in [1] and [10], a straightforward calculation yields the following families of generators:

$$\begin{cases} \lambda(i,j,m) = \phi_{1,2}^m\, \phi_{i,j}\, \phi_{1,2}^{-(m+1)} & \text{for each live } \phi_{i,j} \text{ and for each } m \in \mathbb{Z} \\ \delta(i,j,m) = \phi_{1,2}^m\, \phi_{i,j}\, \phi_{1,2}^{-m} & \text{for each dead } \phi_{i,j} \text{ and for each } m \in \mathbb{Z}. \end{cases}$$

Notice we use $\lambda(i,j,m)$ for each family of generators corresponding to a live automorphism and $\delta(i,j,m)$ for those corresponding to a dead automorphism. Also notice that $\lambda(1,2,m) = 1$ for all $m \in \mathbb{Z}$.

We now obtain the relations for $\ker\theta$ by rewriting the set $\{trt^{-1} \mid t \in T, r \in R\}$ in terms of these generators,where R consists of the relators in Equation 1. For the following, we will denote the live generators of $\mathrm{P}\Sigma_{\mathbf{n}}$ by $l_{i,j}$ and the dead generators of $\mathrm{P}\Sigma_{\mathbf{n}}$ by $d_{i,j}$, i.e.,

$$\phi_{i,j} = \begin{cases} l_{i,j} & \text{if } \theta(\phi_{i,j}) = t \\ d_{i,j} & \text{if } \theta(\phi_{i,j}) = 1. \end{cases}$$

Once again, straightforward calculations yield the following relations for $\ker\theta$; each holds for all $m \in \mathbb{Z}$:

(1) Relations derived from Type C relations of $\mathrm{P}\Sigma_{\mathbf{n}}$.

(a) From each relation of the form $[l_{i,j}, l_{k,j}] = 1$:

$$\lambda(i,j,m)\lambda(k,j,m+1)\lambda(i,j,m+1)^{-1}\lambda(k,j,m)^{-1} = 1$$

(b) From each relation of the form $[l_{i,j}, d_{k,j}] = 1$:

$$\lambda(i,j,m)\delta(k,j,m+1)\lambda(i,j,m)^{-1}\delta(k,j,m)^{-1} = 1$$

(c) From each relation of the form $[d_{i,j}, d_{k,j}] = 1$:

$$\delta(i,j,m)\delta(k,j,m)\delta(i,j,m)^{-1}\delta(k,j,m)^{-1} = 1$$

(2) Relations derived from Type D relations of $\mathrm{P}\Sigma_{\mathbf{n}}$.

(a) From each relation of the form $[l_{i,j}, l_{k,l}] = 1$:

$$\lambda(i,j,m)\lambda(k,l,m+1)\lambda(i,j,m+1)^{-1}\lambda(k,l,m)^{-1} = 1$$

(b) From each relation of the form $[l_{i,j}, d_{k,l}] = 1$:

$$\lambda(i,j,m)\delta(k,l,m+1)\lambda(i,j,m)^{-1}\delta(k,l,m)^{-1} = 1$$

(c) From each relation of the form $[d_{i,j}, d_{k,l}] = 1$:

$$\delta(i,j,m)\delta(k,l,m)\delta(i,j,m)^{-1}\delta(k,l,m)^{-1} = 1$$

(3) Relations derived from Type E relations of $\mathrm{P}\Sigma_{\mathbf{n}}$.

(a) From each relation of the form $[l_{i,j}, l_{i,k}\, l_{j,k}] = 1$:

$$\lambda(i,j,m)\lambda(i,k,m{+}1)\lambda(j,k,m{+}2)\lambda(i,j,m{+}2)^{-1}\lambda(j,k,m{+}1)^{-1}\lambda(i,k,m)^{-1} = 1$$

(b) From each relation of the form $[l_{i,j}, l_{i,k}\, d_{j,k}] = 1$:

$$\lambda(i,j,m)\lambda(i,k,m{+}1)\delta(j,k,m{+}2)\lambda(i,j,m{+}1)^{-1}\delta(j,k,m{+}1)^{-1}\lambda(i,k,m)^{-1} = 1$$

(c) From each relation of the form $[l_{i,j}, d_{i,k}\, l_{j,k}] = 1$:

$$\lambda(i,j,m)\delta(i,k,m)\lambda(j,k,m+1)\lambda(i,j,m+1)^{-1}\lambda(j,k,m)^{-1}\delta(i,k,m)^{-1} = 1$$

(d) From each relation of the form $[l_{i,j}, d_{i,k}\, d_{j,k}] = 1$:

$$\lambda(i,j,m)\delta(i,k,m+1)\delta(j,k,m+1)\lambda(i,j,m)^{-1}\delta(j,k,m)^{-1}\delta(i,k,m)^{-1} = 1$$

(e) From each relation of the form $[d_{i,j}, l_{i,k}\, l_{j,k}] = 1$:

$$\delta(i,j,m)\lambda(i,k,m)\lambda(j,k,m+1)\delta(i,j,m+2)^{-1}\lambda(j,k,m+1)^{-1}\lambda(i,k,m)^{-1} = 1$$

(f) From each relation of the form $[d_{i,j}, l_{i,k}\, d_{j,k}] = 1$:

$$\delta(i,j,m)\lambda(i,k,m)\delta(j,k,m+1)\delta(i,j,m+1)^{-1}\delta(j,k,m+1)^{-1}\lambda(i,k,m)^{-1} = 1$$

(g) From each relation of the form $[d_{i,j}, d_{i,k}\, l_{j,k}] = 1$:

$$\delta(i,j,m)\delta(i,k,m)\lambda(j,k,m)\delta(i,j,m+1)^{-1}\lambda(j,k,m)^{-1}\delta(i,k,m)^{-1} = 1$$

(h) From each relation of the form $[d_{i,j}, d_{i,k}\, d_{j,k}] = 1$:

$$\delta(i,j,m)\delta(i,k,m)\delta(j,k,m)\delta(i,j,m)^{-1}\delta(j,k,m)^{-1}\delta(i,k,m)^{-1} = 1$$

Note that every generator $\phi_{i,j}$ of $\mathrm{P}\Sigma_{\mathbf{n}}$ (except for $\phi_{1,2}$) corresponds to a countable family of generators for $\ker\theta$, and every relation of $\mathrm{P}\Sigma_{\mathbf{n}}$ corresponds to a countable family of relations for $\ker\theta$.

4. The Structure of the Underlying Graphs of $\mathrm{P}\Sigma_{\mathbf{n}}$. Recall [13] that $\mathrm{P}\Sigma_{\mathbf{n}}$ admits a finite presentation with generators

$$\{\phi_{i,j} \mid 1 \le i, j \le n, i \ne j\}$$

and relations given in Equation 1 of Types C, D, and E.

Since the relators of Types C and D are commutators of generators, each group given by removing a subset of the relations containing all Type E relations is a graph group. Each such group has a *defining graph* whose vertices are the generators $\phi_{i,j}$ and whose edges connect (distinct) generators that commute.

Definition 1. Let $C(\mathrm{P}\Sigma_{\mathbf{n}})$ [resp. $D(\mathrm{P}\Sigma_{\mathbf{n}})$] denote the defining graph of the graph group whose generators consist of the $\phi_{i,j}$ and whose relations consist of all Type C [resp. Type D] relations. Similarly, let $CD(\mathrm{P}\Sigma_{\mathbf{n}})$ denote the defining graph of the graph group whose generators consist of the $\phi_{i,j}$ and whose relations consist of all Type C and all Type D relations.

Recall a graph Γ is *k-regular* provided the degree of every vertex in Γ is k. Also recall that the link of $\phi_{i,j}$, $\mathrm{lk}(\phi_{i,j})$, is the full subcomplex generated by all vertices adjacent to $\phi_{i,j}$, while the star of $\phi_{i,j}$, $\mathrm{st}(\phi_{i,j})$, is the cone of $\phi_{i,j}$ over $\mathrm{lk}(\phi_{i,j})$. Throughout this section, since we are only concerned with the underlying graphs, we will take $\mathrm{lk}(\phi_{i,j})$ and $\mathrm{st}(\phi_{i,j})$ to be the 1-skeleta of the topological link and star.

Proposition 1. In $C(\mathrm{P}\Sigma_{\mathbf{n}})$, $\mathrm{st}(\phi_{i,j})$ is a copy of the complete graph K_{n-1}.

Proof. Every pair of vertices of the form $\phi_{k,j}$ and $\phi_{l,j}$ with $1 \leq l, k \leq n$ (l, j, k distinct) are connected by an edge of Type C.

Proposition 2. $C(\mathrm{P}\Sigma_{\mathbf{n}})$ is $(n-2)$-regular, $D(\mathrm{P}\Sigma_{\mathbf{n}})$ is $(n-2)(n-3)$-regular, and $CD(\mathrm{P}\Sigma_{\mathbf{n}})$ is $(n-2)^2$-regular.

Proof. There is an edge of Type C between $\phi_{i,j}$ and $\phi_{k,j}$ if and only if $k \neq i, j$. Thus, $C(\mathrm{P}\Sigma_{\mathbf{n}})$ is $(n-2)$-regular. There is an edge of Type D between $\phi_{i,j}$ and $\phi_{k,l}$ if and only if i, j, k,and l are distinct. Thus, $D(\mathrm{P}\Sigma_{\mathbf{n}})$ is $(n-2)(n-3)$-regular. Since $(n-2)+(n-2)(n-3)$=$(n-2)^2$, it follows that $CD(\mathrm{P}\Sigma_{\mathbf{n}})$ is $(n-2)^2$-regular.

Notice that the proof of Proposition 2 shows that for every i and j, $lk(\phi_{i,j}) = lk(\phi_{j,i})$ in the graph $D(\mathrm{P}\Sigma_{\mathbf{n}})$. Also notice it follows that all links of vertices in $CD(\mathrm{P}\Sigma_{\mathbf{n}})$ [resp. $C(\mathrm{P}\Sigma_{\mathbf{n}})$, $D(\mathrm{P}\Sigma_{\mathbf{n}})$] are isomorphic.

Corollary 1. The link of any vertex of $D(\mathrm{P}\Sigma_{\mathbf{n+2}})$ is isomorphic to $D(\mathrm{P}\Sigma_{\mathbf{n}})$.

Proof. The vertex $\phi_{n+1,n+2}$ is connected to the vertex $\phi_{l,m} (l \neq m)$ by a Type D edge if and only if $1 \leq l \leq n$, $1 \leq m \leq n$, and $l \neq m$. Thus, $\mathrm{lk}\phi_{n+1,n+2} = D(\mathrm{P}\Sigma_{\mathbf{n}})$. The result follows from Proposition 2.

Proposition 3. The link of any vertex of $CD(\mathrm{P}\Sigma_{\mathbf{n+1}})$ is isomorphic to a full subgraph of $CD(\mathrm{P}\Sigma_{\mathbf{n}})$.

Proof. By definition, $\mathrm{lk}(\phi_{n+1,j})$ is a full subgraph of $CD(\mathrm{P}\Sigma_{\mathbf{n+1}})$. Since $\phi_{n+1,j}$ is not adjacent to any vertices of the form $\phi_{n+1,k}$ or $\phi_{k,n+1}$, all vertices take the form $\phi_{l,m}$ where neither l nor m is equal to $n+1$. Thus, $\mathrm{lk}(\phi_{n+1,j})$ is a full subgraph of $CD(\mathrm{P}\Sigma_{\mathbf{n}})$. It follows from Proposition 2 that the link of any vertex of $CD(\mathrm{P}\Sigma_{\mathbf{n+1}})$ is isomorphic to a full subgraph of $CD(\mathrm{P}\Sigma_{\mathbf{n}})$.

Proposition 4. $CD(\mathrm{P}\Sigma_{\mathbf{n+1}})$ is obtained from $CD(\mathrm{P}\Sigma_{\mathbf{n}})$ by adding $2n$ vertices of the form $\phi_{i,n+1}$ and $\phi_{n+1,i}$, together with $3n(n-1)/2$ edges of Type C and $2n(n-1)(n-2)$ edges of Type D.

Proof. Clearly the only vertices that need to be added to $CD(\mathrm{P}\Sigma_{\mathbf{n}})$ are the $2n$ vertices of the form $\phi_{i,n+1}$ and $\phi_{n+1,i}$.

By Proposition 1, for each i $(1 \leq i \leq n)$ there are $n-1$ edges of Type C between $\phi_{n+1,i}$ and the copy of K_{n-1} spanned by the vertices $\phi_{j,i}$ of $CD(\mathrm{P}\Sigma_{\mathbf{n}})$. Since the vertices of the form $\phi_{i,n+1}$ also form a copy of K_n, there are $n(n-1)/2$ edges of Type C connecting these vertices to each other. Thus,

$$n(n-1) + \frac{n(n-1)}{2} = \frac{3n(n-1)}{2}$$

additional edges of Type C are needed when constructing $CD(\mathrm{P}\Sigma_{\mathbf{n+1}})$ from $CD(\mathrm{P}\Sigma_{\mathbf{n}})$. Also, for each i, there are $(n-1)(n-2)$ edges connecting each $\phi_{n+1,i}$ and $\phi_{i,n+1}$ to vertices in $CD(\mathrm{P}\Sigma_{\mathbf{n}})$ of the form $\phi_{j,k}$ with j, k, i, and $n+1$ distinct. Thus, $2n(n-1)(n-2)$ additional edges of Type D are needed when constructing $CD(\mathrm{P}\Sigma_{\mathbf{n+1}})$ from $CD(\mathrm{P}\Sigma_{\mathbf{n}})$.

Definition 2. Let $\phi_{i,j}, \phi_{k,l} \in \mathrm{P}\Sigma_{\mathbf{n}}$. The distance between $\phi_{i,j}$ and $\phi_{k,l}$, denoted $d(\phi_{i,j}, \phi_{k,l})$, is the distance between those two vertices in $CD(\mathrm{P}\Sigma_{\mathbf{n}})$.

Proposition 5. For each $n \geq 4$, the diameter of $CD(\mathrm{P}\Sigma_{\mathbf{n}})$ is 2.

Proof. Let $n \geq 4$, and let $\phi_{i,j} \in CD(\mathrm{P}\Sigma_{\mathbf{n}})$. We first show that each vertex $\phi_{k,l}$ in $CD(\mathrm{P}\Sigma_{\mathbf{n}})$ lies a distance at most 2 from $\phi_{i,j}$. By $\phi_{a,b} \overset{C}{\longleftrightarrow} \phi_{c,d}$ and $\phi_{a,b} \overset{D}{\longleftrightarrow} \phi_{c,d}$, we mean that the vertex $\phi_{a,b}$ is connected to the vertex $\phi_{c,d}$ by an edge of type C and D, respectively. There are six possible forms for $\phi_{k,l}$:

(1) $\phi_{k,j}$ $(k \neq i, j)$. Then $\phi_{i,j} \overset{C}{\longleftrightarrow} \phi_{k,j}$.

(2) $\phi_{k,l}$ $(i, j, k, l$ distinct$)$. Then $\phi_{i,j} \overset{D}{\longleftrightarrow} \phi_{k,l}$.

(3) $\phi_{j,k}$ $(k \neq i, j)$. Then $\phi_{j,k} \overset{C}{\longleftrightarrow} \phi_{l,k} \overset{D}{\longleftrightarrow} \phi_{i,j}$, where $l \neq i, j, k$.

(4) $\phi_{i,k}$ $(k \neq i, j)$. Then $\phi_{i,k} \overset{C}{\longleftrightarrow} \phi_{l,k} \overset{D}{\longleftrightarrow} \phi_{i,j}$, where $l \neq i, j, k$.

(5) $\phi_{k,i}$ $(k \neq i, j)$. Then $\phi_{k,i} \overset{D}{\longleftrightarrow} \phi_{l,j} \overset{C}{\longleftrightarrow} \phi_{i,j}$, where $l \neq i, j, k$.

(6) $\phi_{j,i}$. Then $\phi_{j,i} \overset{D}{\longleftrightarrow} \phi_{l,k} \overset{C}{\longleftrightarrow} \phi_{i,j}$, where i, j, k, l are distinct.

Since $\phi_{i,j}$ and $\phi_{j,i}$ are not adjacent, the diameter of $CD(\mathrm{P}\Sigma_{\mathbf{n}})$ is 2.

5. A Finite Generating Set for $\ker\theta$**.** In [16], Orlandi-Korner computed $\Sigma_1(\mathrm{P}\Sigma_{\mathbf{n}})$, the first BNS-invariant of $\mathrm{P}\Sigma_{\mathbf{n}}$. As an immediate corollary of her main theorem, we have

Corollary 2. (Orlandi-Korner) Let $\theta\colon \mathrm{P}\Sigma_{\mathbf{n}} \to \langle t\rangle$ be as in Section 3. Then $\ker\theta$ is not finitely generated if and only if for some i and j, $\phi_{i,j}$ and $\phi_{j,i}$ are the only live generators under θ.

In this section, we calculate a set of finite generators for $\ker\theta$ when the negation of the above condition holds. Since at least one generator must be live for $\ker\theta \neq \mathrm{P}\Sigma_{\mathbf{n}}$, we will assume $\phi_{1,2}$ is live throughout this section, as in Section 3. (Recall that $\phi_{1,2}$ was chosen as our "transversal generator" in that section. The actual choice of a tranversal generator would depend on which generators are live and dead. Fortunately, the symmetry of McCool's presentation of $\mathrm{P}\Sigma_{\mathbf{n}}$ allows us to renumber accordingly.)

Definition 3. Let $\phi_{i,j}, \phi_{k,l} \in \mathrm{P}\Sigma_{\mathbf{n}}$, with $\phi_{i,j}$ live. By a *live path* from $\phi_{i,j}$ to $\phi_{k,l}$, we mean a path in $\mathrm{CD}(\mathrm{P}\Sigma_{\mathbf{n}})$ in which all intermediate generators are live.

The following is immediate from Lemma 1 of [10].

Corollary 3. (Levy et al.) If there exists a live path from $\phi_{1,2}$ to $\phi_{i,j}$, then the family of generators of $\ker\theta$ corresponding to $\phi_{i,j}$ can be reduced to one generator, namely $\{\phi_{i,j}\}$ or $\{\phi_{i,j}\,\phi_{1,2}^{-1}\}$, depending on whether $\phi_{i,j}$ is dead or live, respectively.

Theorem. Let $\theta\colon \mathrm{P}\Sigma_{\mathbf{n}} \to \langle t\rangle$ be as in Section 3 with $n \geq 4$, and assume it is not true that the only live generators are $\phi_{1,2}$ and $\phi_{2,1}$. Then for each $\phi_{i,j}$, the family of generators of $\ker\theta$ corresponding to $\phi_{i,j}$ can be reduced to at most two generators. Specifically, if $\phi_{i,j}$ is live, it contributes $\{\phi_{i,j}\,\phi_{1,2}^{-1}\}$ or $\{\phi_{i,j}\,\phi_{1,2}^{-1}, \phi_{1,2}\,\phi_{i,j}\,\phi_{1,2}^{-2}\}$, while if $\phi_{i,j}$ is dead, it contributes $\{\phi_{i,j}\}$ or $\{\phi_{i,j},\, \phi_{1,2}\,\phi_{i,j}\,\phi_{1,2}^{-1}\}$.

Proof. Let $\phi_{i,j} \in \mathrm{P}\Sigma_{\mathbf{n}}$. If there exists a live path from $\phi_{1,2}$ to $\phi_{i,j}$, then Corollary 3 applies. It follows that $\phi_{i,j}$ contributes $\{\phi_{i,j}\,\phi_{1,2}^{-1}\}$ to the generating set for $\ker\theta$ if it is live and $\{\phi_{i,j}\}$ if it is dead. On the other hand, if there is no live path from $\phi_{1,2}$ to$\phi_{i,j}$, then $\phi_{i,j}$ does not commute with $\phi_{1,2}$. By Proposition 5, $d(\phi_{1,2}, \phi_{i,j}) = 2$; consequently $\phi_{i,j}$ must be one of four forms: $\phi_{1,k}$, $\phi_{k,1}$, $\phi_{2,k}$ or $\phi_{2,1}$, where k is neither 1 nor 2.

Case I. $\phi_{i,j}$ takes the form $\phi_{1,k}$, where $k \neq 2$.

Subcase IA. $\phi_{1,k}$ is live.

If $\phi_{k,2}$ is live, the relation $[\phi_{1,k}, \phi_{1,2}\,\phi_{k,2}] = 1$ yields the family of relations

$$\lambda(1,k,m)\lambda(1,2,m+1)\lambda(k,2,m+2) = \lambda(1,2,m)\lambda(k,2,m+1)\lambda(1,k,m+2) \quad (2)$$

for all $m \in \mathbb{Z}$ from Section 3, Case 3a. Since $\lambda(1,2,m+1) = \lambda(1,2,m) = 1$ and the relation $[\phi_{1,2}, \phi_{k,2}] = 1$ yields the family $\lambda(k,2,m+2) = \lambda(k,2,m+1) = \lambda(k,2,0)$ (Case 1a), Equation 2 can be rewritten as

$$\lambda(1,k,m+2) = \lambda(k,2,0)^{-1}\lambda(1,k,m)\lambda(k,2,0).$$

Straightforward induction shows

$$\lambda(1,k,m) = \begin{cases} \lambda(k,2,0)^{-m/2}\lambda(1,k,0)\lambda(k,2,0)^{m/2} & m \text{ even} \\ \lambda(k,2,0)^{-(m-1)/2}\lambda(1,k,1)\lambda(k,2,0)^{(m-1)/2} & m \text{ odd.} \end{cases}$$

Hence, $\lambda(1,k,m)$ can be expressed in terms of $\lambda(k,2,0)$ and $\lambda(1,k,0)$, or $\lambda(k,2,0)$ and $\lambda(1,k,1)$. Thus, $\phi_{1,k}$ contributes $\{\phi_{1,k}\,\phi_{1,2}^{-1}, \phi_{1,2}\,\phi_{1,k}\,\phi_{1,2}^{-2}\}$ to the generating set of $\ker\theta$. Similarly, if $\phi_{k,2}$ is dead, $\phi_{1,k}$ contributes $\{\phi_{1,k}\,\phi_{1,2}^{-1}\}$.

Subcase IB. $\phi_{1,k}$ is dead.

If $\phi_{k,2}$ is live, the relation $[\phi_{1,k}, \phi_{1,2}\,\phi_{k,2}] = 1$ yields the family of relations in Case 3e. As in I.A., straightforward induction shows that $\delta(1,k,m)$ can be expressed in terms of $\lambda(k,2,0)$ and $\delta(1,k,0)$, or $\lambda(k,2,0)$ and $\delta(1,k,1)$. As in I.A. it follows that $\phi_{1,k}$ contributes $\{\phi_{1,k}, \phi_{1,2}\,\phi_{1,k}\,\phi_{1,2}^{-1}\}$ to the generating set of $\ker\theta$. Similarly, if $\phi_{k,2}$ is dead, it follows that $\phi_{1,k}$ contributes $\{\phi_{1,k}\}$.

Case II. $\phi_{i,j}$ takes the form $\phi_{k,1}$, where $k \neq 2$. The techniques in this case are essentially identical to those of Case I.

Subcase IIA. $\phi_{k,1}$ is live.

If $\phi_{k,2}$ is live, the relation $[\phi_{k,1}, \phi_{k,2}\,\phi_{1,2}] = 1$ yields the family of relations in Case 3a, which can be used to show $\phi_{k,1}$ contributes $\{\phi_{k,1}\,\phi_{1,2}^{-1}, \phi_{1,2}\,\phi_{k,1}\,\phi_{1,2}^{-2}\}$ to the generating set of $\ker\theta$. Similarly, if $\phi_{k,2}$ is dead,$\phi_{k,1}$ contributes $\{\phi_{k,1}\,\phi_{1,2}^{-1}\}$.

Subcase IIB. $\phi_{k,1}$ is dead.

If $\phi_{k,2}$ is live, the relation $[\phi_{k,1}, \phi_{k,2}\,\phi_{1,2}] = 1$ yields the family of relations in Case 3a, which can be used to show $\phi_{k,2}$ contributes $\{\phi_{k,1}, \phi_{1,2}\,\phi_{k,1}\,\phi_{1,2}^{-1}\}$. Similarly, if $\phi_{k,2}$ is dead, $\phi_{k,1}$ contributes $\{\phi_{k,1}\}$.

Case III. $\phi_{i,j}$ takes the form $\phi_{2,k}$, where $k \neq 1$.

Subcase IIIA. $\phi_{2,k}$ is live.

If $\phi_{1,k}$ is live, the relation $[\phi_{1,2}, \phi_{1,k}\,\phi_{2,k}] = 1$ yields the family of relations in Case 3a. Straightforward induction shows that $\lambda(2,k,m)$ can be expressed in terms of $\lambda(1,k,0)$, or $\lambda(1,k,0)$ and $\lambda(1,k,1)$ (depending on whether $\phi_{k,2}$ is live or dead), and $\lambda(2,k,1)$. In either case, it follows that $\phi_{2,k}$ contributes $\{\phi_{1,2}\,\phi_{2,k}\,\phi_{1,2}^{-2}\}$.

If $\phi_{1,k}$ is dead, the relation $[\phi_{1,2}, \phi_{1,k}\,\phi_{2,k}] = 1$ yields the family of relations in Case 3c. Again, straightforward induction shows that $\lambda(2,k,m)$ can be expressed in terms of $\lambda(2,k,0)$. It follows that $\phi_{2,k}$ contributes $\{\phi_{2,k}\,\phi_{1,2}^{-1}\}$.

Subcase IIIB. $\phi_{2,k}$ is dead.

If $\phi_{1,k}$ is live, the relation $[\phi_{1,k}, \phi_{1,2}\,\phi_{k,2}] = 1$ yields the family of relations in Case 3b. Straightforward induction shows that $\delta(2,k,m)$ can be expressed in terms of $\lambda(1,k,0)$, or $\lambda(1,k,0)$ and $\lambda(1,k,1)$ (depending on whether $\phi_{k,2}$ is live or dead), and $\delta(2,k,1)$. In either case, it follows that $\phi_{2,k}$ contributes $\{\phi_{1,2}\,\phi_{2,k}\,\phi_{1,2}^{-1}\}$.

If $\phi_{1,k}$ is dead, the relation $[\phi_{1,k}, \phi_{1,2}\,\phi_{k,2}] = 1$ yields the family of relations in Case 3d. Straightforward induction shows that $\delta(2,k,m)$ can be expressed in terms of $\delta(1,k,0)$, or $\delta(1,k,0)$ and $\delta(1,k,1)$ (depending on whether $\phi_{k,2}$ is live or dead), and $\delta(2,k,0)$. In either case, it follows that $\phi_{2,k}$ contributes $\{\phi_{2,k}\}$.

Case IV. $\phi_{i,j}$ takes the form $\phi_{2,1}$.

Subcase IVA. $\phi_{2,1}$ is live.

Subcase IVA1. $\phi_{k,1}$ is live.

The relation $[\phi_{k,1}, \phi_{2,1}] = 1$ yields the family of relations in Case 1a. By induction, $\lambda(2,1,m)$ can be expressed in terms of $\lambda(k,2,0)$, $\lambda(k,1,0)$ and$\lambda(2,1,0)$. It follows that $\phi_{2,1}$ contributes $\{\phi_{2,1}\,\phi_{1,2}^{-1}\}$.

Subcase IVA2. $\phi_{2,k}$ is live.

If $\phi_{1,k}$ is live, the relation $[\phi_{2,1}, \phi_{2,k}\,\phi_{1,k}] = 1$ yields the family of relations in Case 3a. By induction, $\lambda(2,1,m)$ can be expressed in terms of $\lambda(2,1,0)$, $\lambda(2,1,1)$, $\lambda(2,k,0)$, $\lambda(1,k,0)$ and $\lambda(k,2,0)$ (Note the results are the same using $\delta(k,2,0)$). It follows that $\phi_{2,1}$ contributes $\{\phi_{2,1}\,\phi_{1,2}^{-1}, \phi_{1,2}\,\phi_{2,1}\,\phi_{1,2}^{-2}\}$. Similarly, if $\phi_{1,k}$ is dead, $\phi_{2,1}$ contributes $\{\phi_{2,1}\,\phi_{1,2}^{-1}\}$.

Subcase IVA3. $\phi_{k,2}$ is live.

If $\phi_{k,1}$ is live, refer to Case IVA1. If $\phi_{k,1}$ is dead, the relation $[\phi_{k,2}, \phi_{k,1}\,\phi_{2,1}] = 1$, yields the family of relations in Case 3e. By induction, $\lambda(2,1,m)$ can be expressed in terms of $\lambda(k,2,0)$, $\delta(k,1,0)$ and $\lambda(2,1,0)$. It follows that $\phi_{2,1}$ contributes $\{\phi_{2,1}\,\phi_{1,2}^{-1}\}$.

Subcase IVA4. $\phi_{1,k}$ is live.

If $\phi_{2,k}$ is live, refer to Case IVA2. If $\phi_{2,k}$ is dead, the relation $[\phi_{2,1}, \phi_{2,k}\,\phi_{1,k}] = 1$, yields the family of relations in Case 3c. By induction, $\lambda(2,1,m)$ can be expressed in terms of $\lambda(2,1,0)$, $\delta(2,k,0)$, $\lambda(1,k,0)$, $\lambda(1,k,1)$ and $\lambda(k,2,0)$. It follows that $\phi_{2,1}$ contributes $\{\phi_{2,1}\,\phi_{1,2}^{-1}\}$.

Subcase IVB. $\phi_{2,1}$ is dead.

Subcase IVB1. $\phi_{k,2}$ is live.

The relation $[\phi_{k,2}, \phi_{k,1}\,\phi_{2,1}] = 1$ yields the family of relations in Case 3d. By induction, $\delta(2,1,m)$ can be expressed in terms of $\lambda(k,2,0)$, $\delta(k,1,0)$ and $\delta(2,1,0)$. It follows that $\phi_{2,1}$ contributes $\{\phi_{2,1}\}$.

Subcase IVB2. $\phi_{k,2}$ is dead, but $\phi_{1,k}$ and $\phi_{2,k}$ are live.

The relation $[\phi_{2,1}, \phi_{2,k}\,\phi_{1,k}] = 1$ yields the family of relations in Case 3e. By induction, $\delta(2,1,m)$ can be expressed in terms of $\lambda(2,k,0)$, $\lambda(1,k,0)$, $\delta(k,2,0)$, $\delta(2,1,0)$, and $\delta(2,1,1)$. It follows that $\phi_{2,1}$ contributes $\{\phi_{2,1}, \phi_{1,2}\,\phi_{2,1}\,\phi_{1,2}^{-1}\}$.

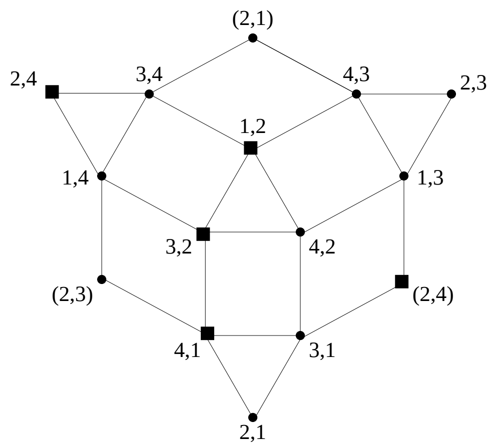
Subcase IVB3. $\phi_{k,2}$ is dead, $\phi_{1,k}$ is live and $\phi_{2,k}$ is dead [resp., $\phi_{1,k}$ is dead and $\phi_{2,k}$ is live].

The relation $[\phi_{2,1}, \phi_{2,k}\,\phi_{1,k}] = 1$ yields the family of relations in Case 3g [resp. Case 3f]. By induction, $\delta(2,1,m)$ can be expressed in terms of $\delta(2,k,0)$ [resp. $\lambda(2,k,0)$], $\lambda(1,k,0)$ [resp. $\delta(1,k,0)$], $\delta(k,2,0)$, and $\delta(2,1,0)$. In either case, it follows that $\phi_{2,1}$ contributes $\{\phi_{2,1}\}$.

Since all possible cases have been considered, it follows that for each $\phi_{i,j}$, the family of generators of $\ker\theta$ corresponding to $\phi_{i,j}$ can be reduced to at most two generators.

Example. Let $\theta\colon \mathrm{P}\Sigma_{\mathbf{4}} \to \langle t\rangle$ be given by $\phi_{1,2}, \phi_{3,2}, \phi_{4,1}, \phi_{2,4} \mapsto t$ and all remaining generators mapped to 1. (Figure 1 contains $CD(\mathrm{P}\Sigma_{\mathbf{4}})$.) Note that $\phi_{i,j}$ is represented as i,j, vertices of the form (i,j) are identified with i,j, live vertices are represented by squares, and dead vertices are represented by circles.

Figure 1



By Corollary 2, $\ker\theta$ is finitely generated. First, consider all generators $\phi_{i,j}$ for which there is a live path from $\phi_{1,2}$ to $\phi_{i,j}$. $\phi_{3,2}$ is live and therefore by Corollary 3 contributes $\{\phi_{3,2}\,\phi_{1,2}^{-1}\}$. Similarly, $\phi_{4,1}$ contributes $\{\phi_{4,1}\,\phi_{1,2}^{-1}\}$. $\phi_{4,2}$ is dead and contributes $\{\phi_{4,2}\}$. Similarly, $\phi_{4,3}$, $\phi_{3,4}$, $\phi_{2,1}$, $\phi_{3,1}$, $\phi_{2,3}$ and $\phi_{1,4}$ correspond to the generators $\{\phi_{4,3}\}$, $\{\phi_{3,4}\}$, $\{\phi_{2,1}\}$, $\{\phi_{3,1}\}$, $\{\phi_{2,3}\}$ and $\{\phi_{1,4}\}$, respectively. Finally, consider all generators with no live path from $\phi_{1,2}$ to the generator, i.e. $\phi_{1,3}$ and $\phi_{2,4}$. By the previous theorem, $\phi_{1,3}$ corresponds to $\{\phi_{1,3},\ \phi_{1,2}\,\phi_{1,3}\,\phi_{1,2}^{-1}\}$. Similarly, $\phi_{2,4}$ contributes $\{\phi_{2,4}\,\phi_{1,2}^{-1}\}$. So, a generating set for $\ker\theta$ is:

$$\{\phi_{3,2}\,\phi_{1,2}^{-1}, \phi_{4,2}, \phi_{4,3}, \phi_{3,4}, \phi_{4,1}\,\phi_{1,2}^{-1}, \phi_{2,1}, \phi_{3,1}, \phi_{2,3}, \phi_{1,4}, \phi_{1,3}, \phi_{1,2}\,\phi_{1,3}\,\phi_{1,2}^{-1}, \phi_{2,4}\,\phi_{1,2}^{-1}\}.$$

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