COMPLEMENTARY INTEGER SEQUENCES THAT HAVE ONLY INITIAL COMMON MOMENTS

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The k^{th} *moment*, $m_k(R)$, of a nonnegative integer sequence $R = \{r_i\}_1^n$ of length n is defined to be the sum of the k^{th} powers of the elements, that is,

$$m_k(R) = \sum_{i=1}^{n} r_i^k.$$

It is convenient to assume that the 0^{th} power of any number is 1. Two equal-length sequences R and Q of nonnegative integers are said to share the k^{th} moment if $m_k(R) = m_k(Q)$. The *common moment set* of R and Q is $P = \{k \mid m_k(R) = m_k(Q)\}$. The *initial interval* of the common moment set is defined to be $P_0 = \{0, 1, 2, \ldots, m(R,Q)\}$, where $m(R,Q) = \max\{j \mid m_k(R) = m_k(Q), 0 \le k \le j\}$. Therefore, the common moment set is $P = P_0 \cup A$, where $A \subset \{m(R,Q)+2, m(R,Q)+3, \ldots\}$. If R and Q are identical sequences, we interpret $p = \infty$. If R and Q are two distinct sequences, the common moment set P is a finite set. We shall discuss nonidentical sequences in this paper.

Chen, Erdős, and Schwenk [2] studied the comment moment sets for the score sequences of complementary tournaments and showed that such a common moment set is $P = \{0, 1, 2, \ldots, 2p\} \cup A$, where $p \ge 0$ and $A \subset \{2p+3, 2p+4, \ldots\}$. Chen [1] provided parallel results for degree sequences of complementary graphs.

Two nonnegative integer sequences $R = \{r_i\}_1^n$ and $Q = \{q_i\}_1^n$ are said to be *complementary* if $r_i + q_i$ is a constant for $i = 1, 2, \ldots, n$. In this paper, we show that the initial interval of the common moment set for complementary sequences is $P = \{0, 1, 2, \ldots, 2p\}$, that is, $m(R,Q) = 2p$ for some $p \ge 0$. We present complementary integer sequences that share only the initial moments. For any given integer $p \ge 0$, we shall construct complementary integer sequences of length 4^p that have the common moment set $P = \{0, 1, 2, \ldots, 2p\}$.

For two sequences of length n both arranged in nonincreasing order, we say that $R = \{r_i\}_1^n$ *dominates* $Q = \{q_i\}_1^n$ if there is an index i_0 such that $r_{i_0} > q_{i_0}$ and $r_i = q_i$ for $i < i_0$. For example, $R = \{6, 5, 3\}$ dominates $Q = \{6, 4, 4\}$. For distinct sequences, one must always dominate the other.

When R dominates Q, we define the *characteristic function* as

$$f(x; R, Q) = \sum_{i=1}^{n} (x^{r_i} - x^{q_i}).$$

We use the standard multiset notation for sequences. Let $\{n_1 \cdot r_1, n_2 \cdot r_2, \ldots, n_p \cdot r_p\}$ denote a sequence consisting of elements r_i with the repetition number n_i, for $i = 1, 2, \ldots, p$.

In [2], Chen, Erdős, and Schwenk gave the following useful Lemma. Chen [1] added the Corollary.

Lemma. Let R and Q be two sequences of length n, and let R dominate Q. Then $f(x; R, Q) = (x-1)^{1+m(R,Q)} g(x)$ with $g(1) \neq 0$.

Proof. By definition, $f(1; R, Q) = 0$, so there exists a $p \geq 0$ such that $f(x; R, Q) = (x-1)^{p+1} g(x)$, with $g(1) \neq 0$. We define $F_k(x)$ recursively by

$$\begin{aligned} &F_0(x) = f(x; R, Q), \\ &F_k(x) = x F'_{k-1}(x), \ \ (k \geq 1). \end{aligned} \tag{1}$$

Evaluation of $F_k(x)$ at $x = 1$ yields

$$F_k(1) = \sum_{i=1}^{n} (r_i^k - q_i^k) = m_k(R) - m_k(Q), \quad \text{for } k \geq 1.$$

Thus, $x - 1$ is a factor of $F_k(x)$ and $F_k(x) = (x-1) g_k(x)$ for $0 \leq k \leq p$. By repeatedly applying (1) to $(x-1)^{p+1} g(x)$, we get

$$F_{p+1}(x) = (x-1) g_{p+1}(x) + (p+1)! x^{p+1} g(x).$$

Therefore, $F_k(1) = 0$ for $k = 0, 1, \ldots, p$, and $F_{p+1}(1) = (p+1)! g(1) \neq 0$. Hence, $p = \max\{\ j \mid m_k(R) = m_k(Q),\ 0 \leq k \leq j\ \} = m(R, Q)$.

Corollary. If $f(x; R, Q) = (x-1)^{p+1} g(x)$ and $g(x)$ is a polynomial with all positive coefficients, then $m_k(R) \neq m_k(Q)$ for $k > p$.

Proof. From the proof of the Lemma, we observe that $F_k(x) = (x-1) g_k(x) + h_k(x)$. If $k > p$, then $h_k(x)$ is a polynomial with all positive coefficients. Therefore, $m_k(R) - m_k(Q) = F_k(1) = h_k(1) > 0$. That is, $m_k(R) \neq m_k(Q)$.

Now we shall present the main results as the following theorems.

Theorem 1. Let R and Q be complementary nonnegative integer sequences. Then $m(R,Q) = 2p$ for some $p \geq 0$.

Proof. Let $R = \{r_i\}_1^n$ and $Q = \{q_i\}_1^n$ with $r_i + q_i = c > 0$ since R and Q are not identical. Then there exists a number sequence $S = \{s_i\}_1^n$ such that $r_i = a + s_i$ and $q_i = a - s_i$ where $a = \frac{1}{2}c$. Hence,

$$r_i^k = (a+s_i)^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} s_i^j,$$

$$q_i^k = (a-s_i)^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} (-1)^j s_i^j.$$

We also use $m_k(S)$ to denote the sum of the k^{th} power of s_i, that is, $m_k(S) = \sum_{i=1}^n s_i^k$. Therefore,

$$m_k(R) = \sum_{i=1}^{n} r_i^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} m_j(S),$$

$$m_k(Q) = \sum_{i=1}^{n} q_i^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} (-1)^j m_j(S). \tag{2}$$

From (2), we observe that $m_k(R) = m_k(Q)$ if and only if

$$2 \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1} a^{k-2j-1} m_{2j+1}(S) = 0. \tag{3}$$

For any k_0, $m(R,Q) \geq k_0$ if and only if $m_k(R) = m_k(Q)$ for $k = 1, 2, \ldots, k_0$. But these k_0 moments are equal if and only if (3) holds for $k = 1, 2, \ldots, k_0$. Thus,

$m_{2j+1}(S) = 0$ for $j = 0, 1, \ldots, \lfloor \frac{k_0-1}{2} \rfloor$. If k_0 is odd, then $\lfloor \frac{k_0-1}{2} \rfloor = \lfloor \frac{(k_0+1)-1}{2} \rfloor$. Therefore, (3) always holds up to an even k, that is, $m(R, Q) = 2p$ for some $p \geq 0$.

Theorem 2. For any integer $p \geq 0$, there exist complementary integer sequences of length 4^p such that the common moment set is $P = \{0, 1, 2, \ldots, 2p\}$.

Proof. We let the two sequences of length 4^p be

$$R = \left\{ \binom{2p+1}{2i} \cdot (2p+1-2i) \mid i = 0, 1, \ldots, p \right\},$$

$$Q = \left\{ \binom{2p+1}{2j+1} \cdot (2p-2j) \mid j = 0, 1, \ldots, p \right\}.$$

Since the sequences are both arranged in nonincreasing order this makes it harder for us to check whether R and Q are complementary. First, we rearrange the sequence Q in nondecreasing order by setting $2i = 2p - 2j$ such that

$$Q = \left\{ \binom{2p+1}{2p+1-2i} \cdot (2i) \mid i = 0, 1, \ldots, p \right\}.$$

Now, the sum of a member in R and its corresponding member in Q is $(2p+1-2i) + (2i) = 2p+1$. Furthermore, $2p+1-2i$ of R and $2i$ of Q have the same repetition number. Therefore, R and Q are complementary sequences. R dominates Q since $2p+1 > 2p$. The characteristic function of R and Q is $f(x; R, Q) = (x-1)^{2p+1}$. By the Lemma and its Corollary, $m_k(R) = m_k(Q)$ for $k \leq 2p$ and $m_k(R) \neq m_k(Q)$ for $k > 2p$, that is, the common moment set $P = \{0, 1, 2, \ldots, 2p\}$.

References

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