AMBIENT SPACES FOR CONES GENERATED BY MATRICIAL INNER PRODUCTS

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Abstract. This paper provides an expository survey of the characterizations and properties of four ambient spaces for cones generated by matricial inner products. The partition induced on the vector space of linear operators on complex square matrices is also displayed.

1. Introduction. Throughout this paper we will use $\mathcal{M}_n = \mathcal{M}_n(\mathbb{C})$ to denote the space of $n \times n$ complex-valued matrices. The real space of $n \times n$ hermitian matrices will be denoted by $\mathcal{H}_n$, and the space of all linear transformations on $\mathcal{M}_n$, by $\mathcal{L}(\mathcal{M}_n)$.

A function $\langle\langle \cdot, \cdot \rangle\rangle \colon \mathcal{M}_n \times \mathcal{M}_n \to \mathcal{M}_n$ defined by $\langle\langle A, B \rangle\rangle = B^*PA + AQB^*$, where P and Q are positive semidefinite and at least one is positive definite provides a vectoral (matricial) inner product [1]. Using this function with various values of P and Q in their Cone Generating Theorem has led Siler and Hill to a lattice of cones [10], which may naturally be thought to reside in any of four real vector spaces. These spaces may be viewed as third–stage generalizations of the real numbers and hermitian matrices in the sense of Barker, Hill, and Haertel [2]. This paper provides a survey of these ambient spaces and their properties.

In Section 2 we summarize information about the spaces $\mathcal{HP}$ (the hermitian–preserving linear transformations), $\mathcal{RP}$ (the real–preserving linear transformations), $\mathcal{SA}$ (the self–adjoint linear transformations), and the set $\mathcal{GH}$ which is a proper subspace of $\mathcal{SA}$. In Section 3 we examine the partition of $\mathcal{L}(\mathcal{M}_n)$ induced by intersections of $\mathcal{HP}$, $\mathcal{RP}$, $\mathcal{SA}$, $\mathcal{GH}$, and their complements.

For $T \in \mathcal{L}(\mathcal{M}_n)$, we will use $\langle T \rangle \in \mathcal{M}_{n^2}$ to denote its representation with respect to basis $\{E_{ij}\}_{i,j=1,\ldots,n} \subseteq \mathcal{M}_n$ ordered antilexicographically. For $A \in \mathcal{M}_{n^2}$, we find it useful to denote its block form as $A = (A_{ij}) = (a_{rs}^{ij}) \in \mathcal{M}_n(\mathcal{M}_n)$, $(i, j, r, s = 1, \ldots, n)$. While all eight of the matrix reorderings of Oxenrider and Hill [7] can be used in our study, for brevity we will restrict our results to Ψ and Γ, defined by $\Psi(A)_{rs}^{ij} = a_{ri}^{sj}$ and $\Gamma(A)_{rs}^{ij} = a_{js}^{ir}$.

2. The Spaces $\mathcal{HP}$, $\mathcal{RP}$, $\mathcal{SA}$, and $\mathcal{GH}$. The real vector space $\mathcal{HP}$ has dimension n^4 with one basis being given by the transformations represented by the set consisting of: the n^2 matrices $E_{ij} \otimes E_{ij}$; the $(n^4 - n^2)/2$ matrices $(E_{ij} + E_{kl}) \otimes (E_{ij} + E_{kl})$; and the $(n^4 - n^2)/2$ matrices $(E_{ij} + \iota E_{kl}) \otimes (E_{ij} - \iota E_{kl})$ for $i, j, k, l = 1, \dots, n$, where the last two sets are indexed such that $(i, j) \neq (k, l)$ and exactly one of $(E_{ij} + E_{kl})$ and $(E_{kl} + E_{ij})$ is chosen for fixed i, j, k, l, and $\iota^2 = -1$; cf. [4].

Proofs of the following characterizations of $\mathcal{HP}$ can be found in [3, 4, 5, 8].

<u>Theorem 2.1</u>. For $T \in \mathcal{L}(\mathcal{M}_n)$, the following are equivalent.

1. $T \in \mathcal{HP}$.
2. $t^{ij}_{rs} = \overline{t^{rs}_{ij}}$, where $\langle T \rangle = \big((t^{rs}_{ij})\big)$.
3. There exist $A_1, \dots, A_s \in \mathcal{M}_n$, and real numbers $\delta_1, \dots, \delta_s$, such that $\langle T \rangle = \sum_{i=1}^{s} \delta_i A_i \otimes \overline{A_i}$, where $s \leq n^2$.
4. There exist $A_1, \dots, A_s \in \mathcal{M}_n$, and $\varepsilon_1, \dots, \varepsilon_s$, that assume the values ± 1, such that $\langle T \rangle = \sum_{i=1}^{s} \varepsilon_i A_i \otimes \overline{A_i}$, where $s \leq n^2$.
5. There exist $A_1, \dots, A_s \in \mathcal{M}_n$, and $(d_{ij}) \in \mathcal{H}_s$, such that $\langle T \rangle = \sum_{i,j=1}^{s} d_{ij} A_i \otimes \overline{A_j}$, where $s \leq n^2$.
6. The block matrix $\big(T(E_{ij})\big)_{1 \leq i,j \leq n}$ is hermitian in $\mathcal{M}_{n^2}$.
7. $\Gamma(\langle T \rangle)$ is hermitian in $\mathcal{M}_{n^2}$.
8. $\Psi(\langle T \rangle)$ is hermitian in $\mathcal{M}_{n^2}$.
9. $\Gamma\big(\langle T \rangle^{tr}\big)$ is hermitian in $\mathcal{M}_{n^2}$.
10. $\Psi\big(\langle T \rangle^{tr}\big)$ is hermitian in $\mathcal{M}_{n^2}$.
11. T is skew–hermitian–preserving.
12. T^* is hermitian–preserving.
13. $\big(T(A)\big)^* = T(A^*)$.
14. There exists $U \in \mathcal{L}(\mathcal{M}_n)$ such that $T(A) = U(A) + (U(A^*))^*$ [5].

The real vector space $\mathcal{RP}$ also has dimension n^4 with one basis being given by the transformations represented by the set consisting of the n^4 matrices $E_{ij} \otimes E_{kl}$, $i, j, k, l = 1, \dots, n$.

The following characterizations of $\mathcal{RP}$ have proofs much like those for $\mathcal{HP}$. In particular, many of them follow immediately from the specialization of $\hat{\kappa}$ to the identity map in Theorem 4.1 of [6].

Theorem 2.2. For $T \in \mathcal{L}(\mathcal{M}_n)$, the following are equivalent.

1. $T \in \mathcal{RP}$.
2. $t_{rs}^{ij} = \overline{t_{rs}^{ij}}$, where $\langle T \rangle = \big((t_{ij}^{rs})\big)$; i.e., $\langle T \rangle \in \mathcal{M}_{n^2}(\mathbb{R})$.
3. There exist real numbers t_{rs}^{ij}, for all $i, j, r, s = 1, \dots, n$, such that $\langle T \rangle = \sum_{i,j,r,s=1}^{n} t_{rs}^{ij} E_{ij} \otimes E_{rs}$.
4. There exist $A_1, \dots, A_s \in \mathcal{M}_n(\mathbb{R})$, and $(d_{ij}) \in \mathcal{M}_s(\mathbb{R})$, such that $\langle T \rangle = \sum_{i,j=1}^{s} d_{ij} A_i \otimes A_j$, where $s \le n^2$.
5. The block matrix $\big(T(E_{ij})\big)_{1 \le i,j \le n}$ is real in $\mathcal{M}_{n^2}$.
6. $\Gamma\big(\langle T \rangle\big)$ is real in $\mathcal{M}_{n^2}$.
7. $\Psi\big(\langle T \rangle\big)$ is real in $\mathcal{M}_{n^2}$.
8. $\Gamma\big(\langle T \rangle^{tr}\big)$ is real in $\mathcal{M}_{n^2}$.
9. $\Psi\big(\langle T \rangle^{tr}\big)$ is real in $\mathcal{M}_{n^2}$.
10. T is skew–real–preserving; i.e., $T\big(\mathcal{M}_n(\iota\mathbb{R})\big) \subseteq \mathcal{M}_n(\iota\mathbb{R})$.
11. T^* is real–preserving.
12. $\overline{\big(T(A)\big)} = T(\overline{A})$.
13. There exist $V_1, \dots, V_s \in \mathcal{M}_n(\mathbb{R})$ and $W_1, \dots, W_s \in \mathcal{M}_n(\mathbb{R})$ such that $T(A) = \sum_{i=1}^{s} V_i A W_i$ for all $A \in \mathcal{M}_n$.
14. There exists $U \in \mathcal{L}(\mathcal{M}_n)$ such that $T(A) = U(A) + \overline{\big(U(\overline{A})\big)}$ [5].

The real vector space of self–adjoint linear transformations, $\mathcal{SA}$, also has dimension n^4 with a basis given by the transformations with representations in the subset of $\mathcal{M}_{n^2}$ consisting of the n^2 matrices E_{jj}; the $(n^4 - n^2)/2$ matrices $E_{jk} + E_{kj}$; and the $(n^4 - n^2)/2$ matrices $\iota E_{jk} - \iota E_{kj}$; $j, k = 1, \dots, n^2$.

For completeness we give the following characterizations of $\mathcal{SA}$.

Theorem 2.3. For $T \in \mathcal{L}(\mathcal{M}_n)$, the following are equivalent.

1. $T \in \mathcal{SA}$.
2. If $\mathcal{B}$ is any orthonormal basis for $\mathcal{M}_n$, then $\langle T \rangle_{\mathcal{B}}$ is hermitian.
3. $\big\langle T(A), B \big\rangle = \big\langle A, T(B) \big\rangle$ for all $A, B \in \mathcal{M}_n$.
4. $\big\langle T(A), A \big\rangle \in \mathbb{R}$ for all $A \in \mathcal{M}_n$.
5. There exist $A_1, \dots, A_s \in \mathcal{H}_n$, and $(d_{ij}) \in \mathcal{M}_s(\mathbb{R})$, such that $\langle T \rangle = \sum_{i,j=1}^{s} d_{ij} A_i \otimes A_j$, where $s \le n^2$.
6. The block matrix $\big(T(E_{ij})\big)_{1 \le i,j \le n}$ represents a hermitian–preserving transformation.

7. $\Gamma\big(\langle T\rangle\big)$ represents a hermitian–preserving transformation.
8. $\Psi\big(\langle T\rangle\big)$ represents a hermitian–preserving transformation.
9. $\Gamma\big(\langle T\rangle^{tr}\big)$ represents a hermitian–preserving transformation.
10. $\Psi\big(\langle T\rangle^{tr}\big)$ represents a hermitian–preserving transformation.

The hermitian matrices can be characterized in terms of inner products as follows: $\mathcal{H}_n = \big\{A \in \mathcal{M}_n \mid \langle Ax, x\rangle \in \mathbb{R} \text{ for all } x \in \mathbb{C}^n\big\}$, where $\langle x, y\rangle = y^*x$. Generalizing this to linear transformations on $\mathcal{M}_n$ leads naturally to the following definition: $\mathcal{GH} = \big\{T \in \mathcal{L}(\mathcal{M}_n) \mid \big\langle\langle T(A), A\rangle\big\rangle \in \mathcal{H}_n \text{ for all } A \in \mathcal{M}_n\big\}$, where $\big\langle\langle X, Y\rangle\big\rangle = Y^*X$ is a matricial inner product [10].

It can easily be verified that $\mathcal{GH}$ is a real vector space of dimension n^2 with a basis given by the transformations represented by the matrices $I \otimes B_{jk}$, where $\big\{B_{jk} \mid j, k = 1, \dots, n\big\}$ is any basis for $\mathcal{H}_n$. (See Theorem 2.4 (6), below.)

Note in the following characterizations taken from [12] that any $T \in \mathcal{GH}$ is completely determined by its action on I. Also, the equivalence of (2) and (3) holds for any matricial inner product [10].

Theorem 2.4. For $T \in \mathcal{L}(\mathcal{M}_n)$, the following are equivalent.

1. $T \in \mathcal{GH}$.
2. $\big\langle\langle T(A), A\rangle\big\rangle \in \mathcal{H}_n$ for all $A \in \mathcal{M}_n$.
3. $\big\langle\langle T(A), B\rangle\big\rangle = \big\langle\langle A, T(B)\rangle\big\rangle$ for all $A, B \in \mathcal{M}_n$.
4. $T(I) \in \mathcal{H}_n$ and $T(A) = T(I)A$ for all $A \in \mathcal{M}_n$.
5. $\langle T\rangle = I \otimes T(I)$, where $T(I) \in \mathcal{H}_n$; i.e., $\langle T\rangle = (T_{ij}) = \big((t^{ij}_{rs})\big)$, where $i, j = 1, \dots, n$, $T_{ij} = O$, where $i \neq j$, $T_{jj} = T(I)$, and for $r, s = 1, \dots, n$, $t^{jj}_{rs} = \overline{t^{jj}_{rs}}$.
6. There exist $A_1, \dots, A_s \in \mathcal{H}_n$, and real numbers $\delta_1, \dots, \delta_s$, such that $\langle T\rangle = \sum_{i=1}^s \delta_i I \otimes A_i$, where $s \leq n^2$.
7. There exist $A_1, \dots, A_s \in \mathcal{H}_n$, and $\varepsilon_1, \dots, \varepsilon_s$, that assume the values ± 1, such that $\langle T\rangle = \sum_{i=1}^s \varepsilon_i I \otimes A_i$, where $s \leq n^2$.
8. $\Psi\big(\langle T\rangle\big) = S \otimes \operatorname{vec}\big(T(I)\big)$, where S is the $1 \times n^2$ matrix with 1 in each of the positions $1, n+2, 2n+3, \dots, n^2$, and 0 elsewhere, and $T(I) \in \mathcal{H}_n$.
9. $\Gamma\big(\langle T\rangle\big) = S^{tr} \otimes \big(\operatorname{vec}\big(T(I)\big)^{tr}\big)^{tr}$, where S is the matrix defined in part 8, and $T(I) \in \mathcal{H}_n$.
10. The block matrix $\big(T(E_{ij})\big) = \big((e_j)^{tr} \otimes T(I)^{(i)}\big)$, where $T(I) \in \mathcal{H}_n$, and $T(I)^{(i)}$ represents the ith column of $T(I)$.

11. $T^* \in \mathcal{GH}$.
12. There exists $H \in \mathcal{H}_n$ such that $T(A) = HA$ for all $A \in \mathcal{M}_n$.

It can easily be verified that $\mathcal{GH}$ is neither closed under composition nor under $*$–congruence (conjuctivity). For completeness, we list a few more properties from [12].

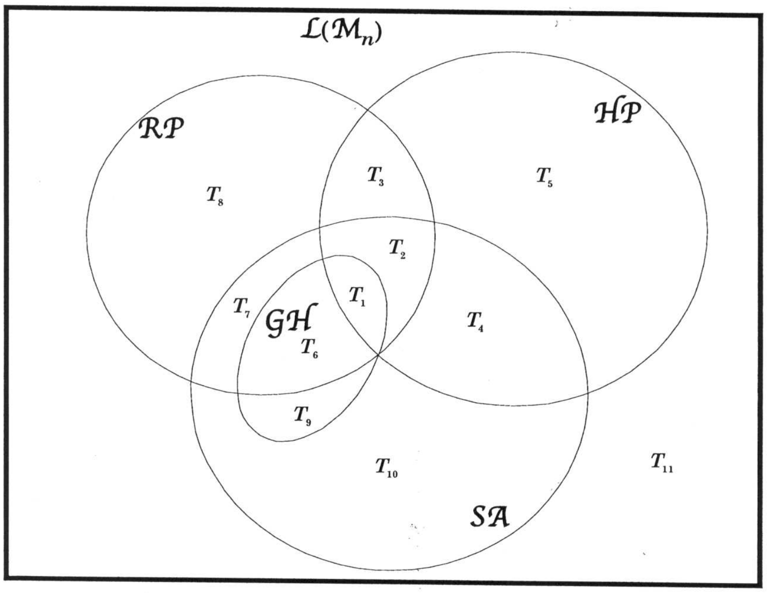
Theorem 2.5. For $T \in \mathcal{GH}$:

1. $\langle T\rangle = I \otimes T(I)$.
2. $\langle T\rangle$ is unitarily diagonalizable.
3. If the distinct eigenvalues of $T(I)$ are $\lambda_1, \dots, \lambda_s$, with algebraic multiplicities $p_1, \dots, p_s$, respectively, and $x_1, \dots, x_n$ is a complete set of orthonormal eigenvectors of $T(I)$, then the distinct eigenvalues of $\langle T\rangle$ are $\lambda_1, \dots, \lambda_s$ with algebraic multiplicities $np_1, \dots, np_s$, respectively, and $\{e_j \otimes x_k \mid j, k = 1, \dots, n\}$ is a complete set of orthonormal eigenvectors of $\langle T\rangle$.

Theorem 2.6. If $T \in \mathcal{GH}$, and $T(I)$ is nonsingular, then for any $A \in \mathcal{M}_n$,

1. there exists a constant, c, such that $\det\big(T(A)\big) = c\det(A)$ and
2. $\operatorname{rank}\big(T(A)\big) = \operatorname{rank}(A)$.

3. Interrelationships. While there are 16 possible intersections of the four real spaces $\mathcal{HP}$, $\mathcal{RP}$, $\mathcal{SA}$, and $\mathcal{GA}$ and their (set) complements in $\mathcal{L}(\mathcal{M}_n)$, only 11 of these intersections are nonempty. The following Venn diagram shows the partition induced by these spaces with the representative transformations from each piece defined below. We assume that $n \geq 2$ since for $n = 1$, $\mathcal{HP} = \mathcal{RP} = \mathcal{SA} = \mathcal{GH}$ are all simply the set of real numbers.



Definitions of Representative Transformations.

$T_1(A) = A$.

$T_2(A) = \mathrm{tr}(A)I$, where $\mathrm{tr}(A)$ denotes the trace of A.

$T_3(A) = E_{11}AE_{12} + E_{21}AE_{11}$.

$T_4(A) = \iota E_{21}AE_{21} - \iota E_{12}AE_{12}$.

$T_5(A) = \iota E_{22}AE_{11} - \iota E_{11}AE_{22}$.

$T_6(A) = E_{11}A$.

$T_7(A) = E_{12}AE_{11} + E_{21}AE_{11}$.

$T_8(A) = E_{11}AE_{11} + 3E_{12}AE_{11}$.

$T_9(A) = \iota E_{12}A - \iota E_{21}A$.

$T_{10}(A) = (2+\iota)E_{12}AE_{11} + (2-\iota)E_{21}AE_{11}$.

$T_{11}(A) = \iota A$.

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