ON POSITIVE DERIVATIVES AND MONOTONICITY

Stephen D. Casey and Richard Holzsager

Abstract. A fundamental result of calculus states that a function f with a positive derivative Df on an interval is increasing on that interval. This result follows directly from the Mean Value Theorem. We explore the extent to which the hypotheses of the Mean Value Theorem can be weakened and f still shown to be increasing. Let f be a continuous function on an interval $[a, b]$. By constructing counterexamples using Cantor–Lebesgue functions, we show that the assumption $Df > 0$ *a.e.* does not imply that f is increasing. We can, however, show that if $Df > 0$ except on a countable subset of an interval, then f is increasing. We call this the Countable Exceptional Set Theorem. This theorem is generalized by the Goldowsky–Tonelli Theorem, which tells us that if Df exists except on a countable subset of an interval and $Df > 0$ *a.e.*, then f is increasing. We then show that if the exceptional set S is uncountable, then we can construct a continuous function f for which $Df > 0$ on $[a, b]\backslash S$, but for which $f(a) > f(b)$. The key item is that the exceptional set contains a perfect set. Moreover, this occurs whenever the exceptional set is uncountable. Thus, in a very natural sense, Goldowsky–Tonelli is a vacuous extention of the Countable Exceptional Set Theorem.

1. Introduction. One of the most fundamental and useful results from calculus is that if a function f is differentiable with positive derivative Df on an interval, then it is increasing there. If we assume f is continuous on $[a, b]$ and differentiable on (a, b), then the result follows directly from the Mean Value Theorem. The purpose of this paper is to ask about generalizations of this result. In particular, we want to obtain, given certain assumptions, the most straightforward yet most general theorem possible. We will assume throughout the paper that all of the functions considered in connection to our investigation are continuous. If we do not assume continuity we can vary behavior at a single point and thus change monotonicity.

The paper is structured around a dialog between the two authors, which boiled down to answering two questions. The first question we asked was the following.

> If we weaken the hypotheses of the Mean Value Theorem, does $Df > 0$ imply that f is increasing?

As one would expect, the answer to our first question is no. To see this, we assume, in addition to continuity, only that $Df > 0$ almost everywhere (*a.e.*), i.e., except on a set of Lebesgue measure zero. (The texts of Apostol [2], Boas [4], Royden [18], Rudin [19, 20], or Wheeden and Zygmund [23] can be used as references for background information for this paper.) This condition is too weak, as is shown by the following example. Let $\mathcal{C}$ denote the Cantor–Lebesgue function associated with the Cantor middle–thirds set. Let

$$f(x) = \frac{x}{2} + \mathcal{C}(1-x).$$

Then f is a continuous mapping on $[0,1]$ with

$$Df = \frac{1}{2} \ a.e.$$

However,

$$f(0) = 1 > \frac{1}{2} = f(1).$$

The function f is not increasing. Here, the *risers* of the Cantor–Lebesgue function, which occur over the Cantor middle–thirds set, allow the function f to *flow* against the derivative. Measure zero gives too much room allowing for this flow. Other examples include

$$f_\alpha(x) = \alpha x + \mathcal{C}(1-x) \quad \text{and} \quad g_\alpha(x) = \alpha x - \mathcal{C}(x), \alpha \in (0,1)$$

(see Figures 1 and 2). Moreover, these examples can be generalized. Given that every Cantor set is homeomorphic to the Cantor middle–thirds set [12], we can construct an associated Cantor–Lebesgue function, which is differentiable with zero derivative on the complement of the set. Use this Cantor–Lebesgue function to construct the example, as was done above.

This led to our second question.

> Assuming that f is continuous on $[a,b]$ and differentiable on (a,b) *a.e.*, what additional conditions can we impose on Df to guarantee that f is monotone increasing?

We can formulate this second question in terms of sets.

<u>Definition 1</u>. Let f be continuous on $[a,b]$ and suppose that $Df > 0$ on $[a,b]\backslash S$. Then S is called the *exceptional set*.

What conditions can we put on S to guarantee that f is increasing? If we assume that the exceptional set is countable, we can prove the following (Section 2.1). We refer to this as the *Countable Exceptional Set Theorem*.

Theorem 1. Let f be a continuous function on $[a, b]$ and suppose that $Df > 0$ on $[a, b]\backslash S$, where S is countable. Then $f(a) < f(b)$.

A reading of Saks' classic treatise *Theory of the Integral* [21] gives that this result is generalized by the Goldowsky–Tonelli Theorem. (Also see [13].)

Theorem 2. (Goldowsky–Tonelli [9, 13, 21, 22]). Let f be a continuous function on $[a, b]$ and suppose that Df exists (finite or infinite) on $[a, b]\backslash S$ where S is countable. Also suppose that $Df \geq 0$ *a.e.* on $[a, b]$. Then f is a non–decreasing function on $[a, b]$.

We give a proof of Goldowsky–Tonelli in the spirit of our paper in Section 2.2. This still, however, does not answer our second question.

We give the answer in Section 3. We show that if the exceptional set S includes a Cantor set, then we can construct a continuous function f for which $Df > 0$ on $[a, b]\backslash S$, but for which $f(a) > f(b)$ (Theorem 4). Moreover, this occurs whenever the exceptional set is uncountable (Theroems 5 and 6). The key item is that whenever the exceptional set is uncountable, it contains a perfect set, and therefore contains a Cantor set. This in turn shows that Goldowsky–Tonelli is a vacuous extension of the Countable Exceptional Set Theorem, in that there are no additional functions to which Goldowsky–Tonelli applies but Countable Exception does not apply. Goldowsky–Tonelli, however, requires less information. Given a function satisfying the hypotheses of Goldowsky–Tonelli, one needs Goldowsky–Tonelli to show that it satisfies the hypotheses of Countable Exception.

There are generalizations of Goldowsky–Tonelli. In particular, there are theorems of Tolstoff and Zahorski [5]. Both of these weaken the condition that f is continuous. In fact, by introducing generalizations of continuity and differentiability, a variety of theorems can be derived. An excellent discussion of these can be found in Chapter XI of [5].

Note that general Cantor sets can have positive measure ("fat Cantor sets" [18, 23]) or exist on *finer* sets (in the sense of dimension) than the middle–third sets. All Cantor sets are, however, uncountable.

2. Countability is Sufficient.

2.1. The Countable Exceptional Set Theorem. We now give a proof of Theorem 1. We assume that f is a real–valued function which is continuous on $[a,b]$, and suppose that $Df > 0$ on $(a,b)\backslash S$, where S is countable.

Proof. We first show that $f(a) \leq f(b)$. Suppose, by way of contradiction, that $f(a) > f(b)$. Choose γ such that $f(a) > \gamma > f(b)$, with $\gamma \notin f(S)$. This is possible since $f(S)$ is countable, and therefore does not exhaust $(f(b), f(a))$. Let

$$A = \{x \in [a,b] : f(x) > \gamma\}.$$

Since $a \in A$, $A \neq \emptyset$. Therefore, A has a supremum. Denote it by c. Then

(i) if $x > c$, $f(x) \leq \gamma$,
(ii) for each $\delta > 0$ there exists x_δ such that $c - \delta < x_\delta \leq c$ and $f(x_\delta) > \gamma$.

Now, suppose $f(c) > \gamma$. Then, since f is continuous, $f(c+\epsilon) > \gamma$ for sufficiently small $\epsilon > 0$, contradicting (i). On the other hand, if $f(c) < \gamma$, $f(c-\epsilon) < \gamma$ for sufficiently small $\epsilon > 0$, which contradicts (ii). Therefore, $f(c) = \gamma$. From (i) and (ii), we have that $Df(c) \leq 0$, contradicting the assumption that $Df > 0$ on $(a,b)\backslash S$. Thus, $f(b) \geq f(a)$.

To complete the proof, let $x \in (a,b)$, and apply the results derived above on $[a,x]$, $[x,b]$, obtaining $f(a) \leq f(x) \leq f(b)$. If $f(b) = f(a)$, f is constant on $[a,b]$ and so $Df = 0$ on $[a,b]$, contradicting $Df > 0$. Thus, $f(b) > f(a)$.

Remarks.

• Since the result is true on any $[\alpha,\beta]$, $a \leq \alpha < \beta \leq b$, f is strictly increasing on $[a,b]$.

• The proof also works if we only assume that the Dini derivatives satisfy $D^+f > 0$, $D_+f > 0$, or $D_-f > 0$. To include the case where $D^-f > 0$(the upper left Dini derivative), consider $c' = \inf\{x \in [a,b] : f(x), \gamma\}$. Observe that $D^-f(c') \leq 0$.

• The proof is a generalization of an argument of Bers [3]. Also see the adjacent paper of Cohen [7].

• On page 153 of [8] there is a result similar to Theorem 1 in that it allows a countable set of exceptions, that is, a countable set of points at which Df is not necessarily positive. Also see the monograph of Boas [4].

We also get the following proposition, which is an exercise in Royden [18], as a corollary.

Proposition 1. If f is continuous on $[a, b]$ and one of its derivatives is everywhere non–negative on (a, b), then f is non–decreasing on $[a, b]$.

2.2. The Goldowsky–Tonelli Theorem. We prove the following, which also gives Goldowsky–Tonelli.

Theorem 3. If f is continuous on $[a, b]$ and if the lower right Dini derivative D_+f is greater than or equal to 0 except on a set of measure 0 and greater than $-\infty$ except on a countable set, then f is non–decreasing.

Proof. Let μ denote Lebesgue measure, let $a_1, a_2, \ldots$ be the points in $[a, b)$ where the lower right derivative is $-\infty$, and let A_n be the measure zero subset of $[a, b)$ where the derivative is greater than $-n$. Since A_n has measure zero, given any $\delta_n > 0$, there exists an open set U_n that contains A_n with measure $\mu(U_n) < \delta_n$. We can choose δ_n to be arbitrarily small. For each $x \in A_n$ we can find an interval $[x, t_x)$ contained in U_n such that

$$f(t_x) - f(x) > -(n + 1)(t_x - x) > -(n + 1)\delta_n.$$

For each $x = a_n$, given any $\epsilon_n > 0$, we can find an interval $[x, t_x)$ with

$$f(t_x) - f(x) > -\epsilon_n.$$

Again, we can choose ϵ_n to be arbitrarily small. For every remaining x in $[a, b)$, since $D_+f(x) \geq 0$, we can choose any $\eta > 0$ and find $t_x > x$ such that

$$f(t_x) - f(x) > -\eta(t_x - x).$$

Therefore, by Zorn's Lemma [15], we can find a countable family of intervals $[x, t_x)$ such that $[a, b) = \bigcup[x, t_x)$. Thus, $f(b) - f(a) = \sum f(t_x) - f(x)$ and $b - a = \sum t_x - x$, since both series telescope. Using these two telescoping series and the three inequalities above, we get

$$f(b) - f(a) = \sum f(t_x) - f(x) > -\sum(n + 1)\delta_n - \sum \epsilon_n - \eta \sum t_x - x$$

$$= -\sum(n + 1)\delta_n - \sum \epsilon_n - \eta(b - a).$$

By proper choice of δ_n, ϵ_n, and η, we can make this as close to 0 as we wish by showing that $f(b) \geq f(a)$. Since the same applies to $[a, c)$ for any $c \in (a, b)$, this proves the theorem.

Corollary 1. If f is continuous on $[a,b]$, has right derivative 0 except on a set of measure 0, and has finite right Dini derivatives except at countably many points, then f is constant.

Proof. Apply the above to both f and $-f$.

Corollary 2. If f is differentiable on an interval $[a,b]$ and has zero derivative except on a set of measure 0, then f is constant.

3. Between Countable and Measure Zero. The results in the previous sections naturally lead us to the second question we asked in the introduction. Namely, if we assume that f is continuous on $[a,b]$ and differentiable on (a,b) *a.e.*, then what additional conditions can we impose on Df which guarantees that f is monotone increasing? Is there any condition between countable and measure zero that will guarantee monotonicity?

Note that if our exceptional set contains a Cantor set, then we can construct a continuous non–monotone function g such that $Dg > 0$ *a.e.* Simply compose the homeomorphism from the Cantor set onto the middle–thirds set with the function f given in Section 1.

Theorem 4. If g is continuous on $[a,b]$ and the exceptional set S of points where Dg is not positive (or fails to exist) does not contain a perfect set, then g is non–decreasing. However, for any exceptional set that does contain a perfect set, there are counterexamples.

Proof. Let g be a continuous function on some interval containing $[a,b]$ with exceptional set S. Assume that $g(a) > g(b)$. We shall show that S contains a perfect set. Define the function h on $[a,b]$ by

$$h(x) = \sup\{g(t) : x \leq t \leq b\}.$$

For any x with $Dg(x) > 0$, $g(x) < h(x)$, such an x lies in one of the intervals where h is constant (see Figure 3). Therefore, we have a closed set, which must be uncountable because h maps it onto $[g(b), g(a)]$. Since we have an uncountable closed set, any such set can easily be shown to contain a perfect set, which in turn contains a Cantor set. (We can alternatively argue as follows. Consider the set of intervals where h is constant. By constructing a Cantor set of each of these intervals, we can create the desired set.) Homeomorphically map this set onto the Cantor middle–thirds set. Composing with the function f of Section 1 gives us the function we need.

It would appear that this provides an answer to our second question. However, there is more for us to show. Recall that the class of *Borel sets* is the smallest family containing the intervals that is closed under the operations of countable union and countable intersection [18, 23].

Theorem 5. The set of points where a continuous function fails to have a positive derivative is a Borel set.

Proof. The condition that $Dg(x)$ exist and be positive can be stated as follows. For any integer $n > 0$ there is an integer $m > 0$ and there are rational numbers $0 < r < s$ with $(s - r) < 1/n$ such that for any rational $t \neq 0$ with $|t| < 1/m$, $r < (g(x+t) - g(x))/t < s$. Thus, for any positive rational numbers $r < s$, and any rational $t \neq 0$, we let $U_{rst} = \{x : r < (g(x+t) - g(x))/t < s\}$. The set in question is

$$\bigcap_n \bigcup_m \bigcup_{((s-r)<1/n)} \bigcap_{(|t|<1/m)} U_{rst}.$$

Theorem 6. Any uncountable Borel set contains a perfect set.

Proof. A proof can be found in Section 32 of [10].

Remark. The result is due independently to Alexandroff [1] and Hausdorff [11].

Thus, if the exceptional set is countable, the function is increasing. However, if it is uncountable, we can construct f such that $Df > 0$ *a.e.*, but for which $f(a) > f(b)$. There is no condition between countable and measure zero that guarantees monotonicity. This in turn shows that Goldowsky–Tonelli is a vacuous extension of the Countable Exceptional Set Theorem, in that there are not additional functions to which Goldowsky–Tonelli applies but Countable Exception does not apply. Goldowsky–Tonelli, however, requires less information. Given a function satisfying the hypotheses of Goldowsky–Tonelli, one needs Goldowsky–Tonelli to show that it satisfies the hypotheses of Countable Exception.

4. Must the Derivative Be Strictly Positive? Since we have seen to what extent positive derivative implies monotonicity, it is natural for us to ask to what extent monotonicity implies positive derivative. We close with a few results. (The reader may also be interested in the paper of Katznelson and Stromberg [14].)

Theorem 7. **(Riesz)** There is a continuous strictly increasing function that has zero derivative almost everywhere.

This if from Section 24 of Riesz and Nagy [17]. Start with $f(0) = 0$, $f(1) = 1$. Define $f(1/2) = (f(0)+2f(1))/3$, $f(1/4) = (f(0)+2f(1/2))/3$, $f(3/4) = (f(1/2)+$

$2f(1))/3$, and so forth, defining f at every dyadic fraction as a weighted sum. The function extends continuously to $[0, 1]$. You can picture this as a linear function which is successively distorted by raising the graph at the midpoints of successively smaller dyadic intervals. Figure 4 shows the final result. The derivative at any point, if it exists, is the limit of a product of factors, each of which is either 2/3 or 4/3. But the only number that can be represented as such a limit is 0. Since Lebesgue's Theorem says the derivative must exist except on a set of measure 0, it must therefore be 0 except on a set of measure 0.

Recall that if we require differentiability, the situation is different. We have shown that if f is differentiable on an interval $[a, b]$ and has zero derivative except on a set of measure 0, then f is constant. On the other hand, if we just want the set where the derivative is non–zero to be small, we get a very different result.

Theorem 8. For any ϵ such that $0 < \epsilon < (b - a)$, there is a $\mathcal{C}^\infty$ function on $[a, b]$ that is strictly increasing, but has zero derivative except on a set of measure at most ϵ.

Proof. Choose a sequence $\{a_n\}$ of positive numbers summing to ϵ, where $0 < \epsilon < (b - a)$. Now perform a "middle–third" construction, removing an open interval of length a_1 from the interior of $[a, b]$, then an interval from the interior of each of the two remaining intervals with lengths adding up to a_2, then intervals from the four remaining ones with lengths adding up to a_3, etc., being sure to choose the intervals we remove so that the remaining intervals have lengths that approach 0 (e.g., choose each interval to be removed so that the two remaining intervals have equal length). Now define a function f which is 0 on the remaining set, which clearly has measure $1 - \epsilon$, and which satisfies

$$f(x) = \exp(-1/(x - c)^2 - 1/(x - d)^2)$$

on each removed interval (c, d). This function is $\mathcal{C}^\infty$. Because the lengths of the intervals "left behind" approach 0, f is not identically 0 on any interval within $[a, b]$. An indefinite integral of f is the function we seek.

We close by noting that if we assume simple differentiability, then we have a (somewhat complicated) variation of the previous example with zero derivative on a dense subset of measure at least $1 - \epsilon$.

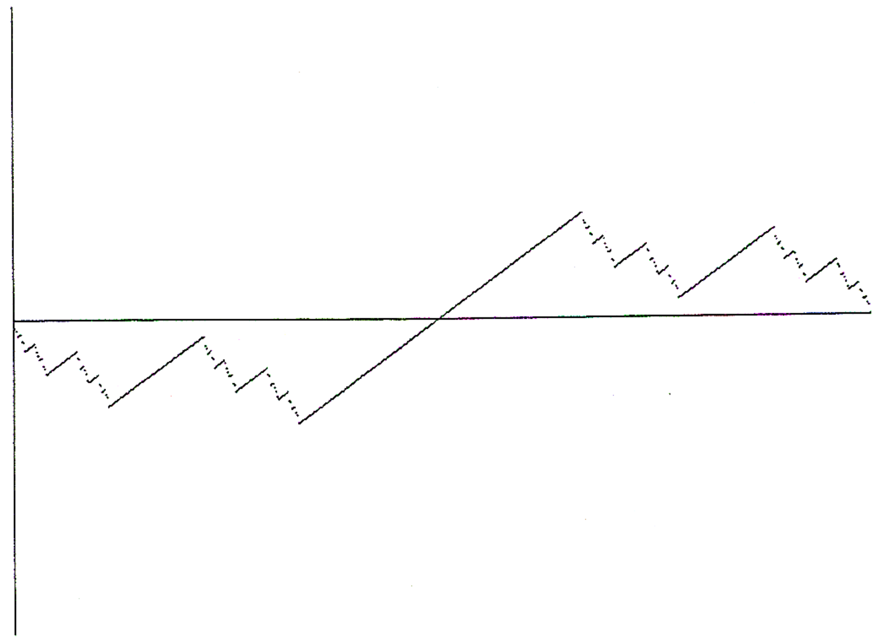

Figure 1. The function $g_{1/2}(x)$

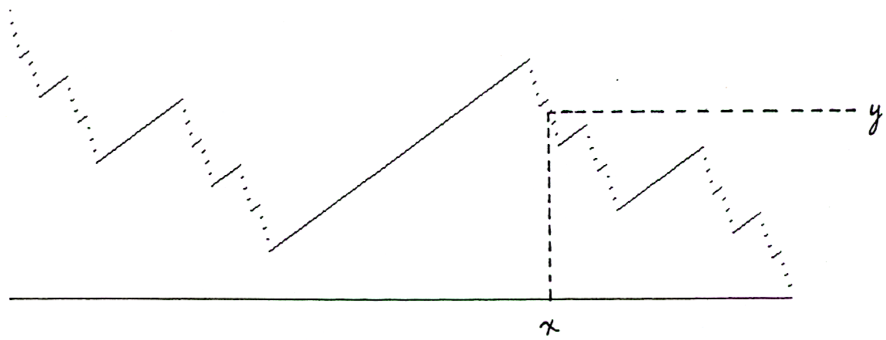

Figure 2. Magnification of a part of the graph of $g_{1/2}$

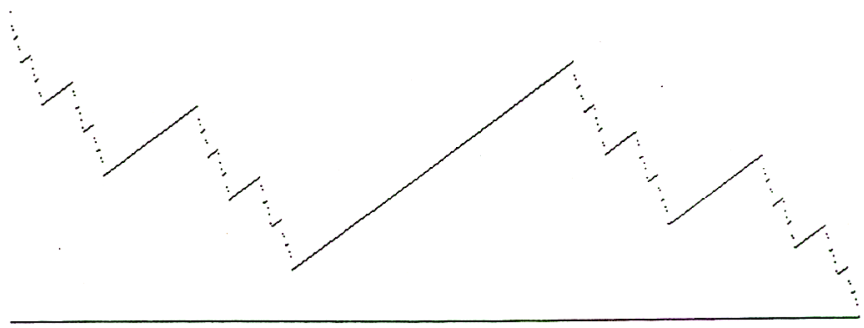

Figure 3. (A.) For Theorem 4 — The function g

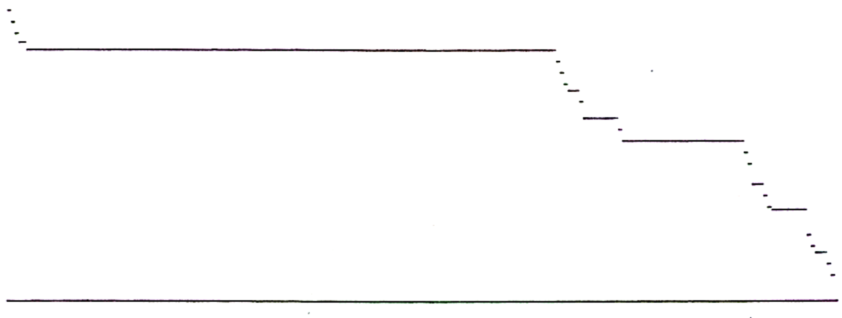

Figure 3. (B.) For Theorem 4 — The function h

Figure 4. Function described in Riesz–Nagy

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Stephen D. Casey
Department of Mathematics and Statistics
American University
4400 Massachusetts Ave., N.W.
Washington, DC 20016–8050
email: scasey@american.edu

Richard Holzsager
Department of Mathematics and Statistics
American University
4400 Massachusetts Ave., N.W.
Washington, DC 20016-8050
email: holzsagr@american.edu