EVALUATION OF A CLASS OF MULTIPLE INTEGRALS

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Abstract. In his paper [1], K. L. D. Gunawardena introduced two sequences of multiple integrals (see (1.1) and (1.2) below) encountered in one of his courses of mathematical statistics. Using a key probability result, he evaluated these integrals. At the end of [1], he asked the question whether these integrals could be computed without using results from probability theory. The aim of this paper is to answer that question and make direct computations of these integrals by elementary methods without using probabilistic tools.

1. Introduction. In his paper [1], K. L. D. Gunawardena has considered the following sequences of multiple integrals

$$I_n := \int_0^1 \cdots \int_0^1 \frac{\theta^n (x_1 \cdots x_n)^{\theta-1}}{\ln(x_1 \cdots x_n)}\, dx_1 \cdots dx_n \tag{1.1}$$

and

$$J_n := \int_0^\infty \cdots \int_0^\infty \frac{\theta^n [(1+x_1)\cdots(1+x_n)]^{-(\theta+1)}}{\ln[(1+x_1)\cdots(1+x_n)]}\, dx_1 \cdots dx_n \tag{1.2}$$

where $\theta > 0$ and n is an integer greater than or equal to 2. He computed separately, (1.1) and (1.2), by using a key probability result and asked whether (1.1) and (1.2) could be computed without making use of results from probability theory.

The purpose of this note is to make direct computations of (1.1) and (1.2) by using tools only from classical analysis. More precisely, we compute (1.1) and (1.2) by using some convenient and natural changes of variables. Before starting computations, we need to introduce the following sequence of multiple integrals:

$$K_n := \int_0^\infty \cdots \int_0^\infty \frac{\theta^n \exp\left(-\theta(y_1 + \cdots + y_n)\right)}{y_1 + \cdots + y_n}\, dy_1 \cdots dy_n, \tag{1.3}$$

for all integers n greater than or equal to 2. The introduction of these integrals is justified by the following observation.

Observation. For all integers n greater than or equal to 2, we have the following equalities:

$$-I_n = J_n = K_n. \tag{1.4}$$

To prove the first equality in (1.4), it is sufficient to use the following change of variable:

$$y_i := \frac{1}{1+x_i}, \quad \text{for all} \quad i = 1, 2, ..., n. \tag{1.5}$$

To prove the second equality in (1.4), it is sufficient to use the following change of variable:

$$y_i := \ln(1+x_i), \quad \text{for all} \quad i = 1, 2, ..., n. \tag{1.6}$$

By this observation, we are led to compute only the integrals K_n for $n \geq 2$. It is clear that K_1 is infinite. To compute K_n, we shall use the spherical coordinates in the Euclidean space $\mathbb{R}^n$.

2. Recalls on Spherical Coordinates. In the Euclidean space $\mathbb{R}^n$, the spherical coordinates $(r, \phi_1, \ldots, \phi_{n-1})$ correponding to the point $(x_1, \ldots, x_n)$ are given by

$$\begin{aligned}
x_1 &= r \sin\phi_{n-1} \cdots \sin\phi_3 \sin\phi_2 \sin\phi_1 \\
x_2 &= r \sin\phi_{n-1} \cdots \sin\phi_3 \sin\phi_2 \cos\phi_1 \\
x_3 &= r \sin\phi_{n-1} \cdots \sin\phi_3 \cos\phi_2 \\
\cdots &= \cdots\cdots\cdots\cdots \\
x_{n-1} &= r \sin\phi_{n-1} \cos\phi_{n-2} \\
x_n &= r \cos\phi_{n-1},
\end{aligned} \tag{2.1}$$

where $0 \leq r < \infty$; $0 \leq \phi_1 < 2\pi$; and $0 \leq \phi_k < \pi$ for all $k = 2, \ldots n-1$. Conversely, for all $k = 1, 2, \ldots, n-1$, by setting $r_k^2 := x_1^2 + \cdots + x_k^2$ and $r_n := r$, we have

$$\cos(\phi_k) = \frac{x_{k+1}}{r_{k+1}}, \tag{2.2}$$

and

$$\sin(\phi_k) = \frac{r_k}{r_{k+1}}. \tag{2.3}$$

We recall that the Jacobian of this change of variables is given by

$$r^{n-1}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\cdots\sin\phi_2. \tag{2.4}$$

Let $\mathbb{S}_n$ be the unit sphere (i.e., the set of $x=(x_1,...,x_n)$ with $\|x\|=1$, where $\|.\|$ is the usual Euclidean norm of $\mathbb{R}^n$). Then (2.4) means that for any Lebesgue integrable function f on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \left[\int_{\mathbb{S}_n} f(ru)\,d\sigma(u)\right] r^{n-1}\,dr, \tag{2.5}$$

where $x=ru\in\mathbb{R}^n=[0,\infty)\times\mathbb{S}_n$ and

$$d\sigma(u)=\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\cdots\sin\phi_2\,d\phi_1\,\cdots\,d\phi_{n-2}\,d\phi_{n-1} \tag{2.6}$$

is the restriction of the Lebesgue measure on $\mathbb{S}_n$.

Let us consider the set $\mathbb{R}^n_+ := \{x=(x_1,\dots,x_n)\in\mathbb{R}^n : 0\le x_i\le\infty,$ for all $i=1,\dots,n\}$, and denote

$$\mathbb{S}^n_+ := \{u=(1,\phi_1,\cdots,\phi_{n-1})\in\mathbb{S}_n : 0\le\phi_i\le\frac{\pi}{2},\ \text{for all } i=1,\dots,n-1\}.$$

Then, it is clear that for all $x=ru\in\mathbb{R}^n$, we have

$$x\in\mathbb{R}^n_+ \Longleftrightarrow u\in\mathbb{S}^n_+, \tag{2.7}$$

and for every Lebesgue integrable function f on $\mathbb{R}^n_+$, we have

$$\int_{\mathbb{R}^n_+} f(x)\,dx = \int_0^\infty \left[\int_{\mathbb{S}^n_+} f(ru)\,d\sigma(u)\right] r^{n-1}\,dr. \tag{2.8}$$

3. Computation of $\mathbf{K_n}$, $\mathbf{I_n}$, and $\mathbf{J_n}$. First, we start by setting $y_i := {x_i}^2$ where $0\le x_i<\infty$ for all $i=1,2,\dots,n$. With these variables, we have

$$dy_1\cdots dy_n = 2^n x_1\cdots x_n dx_1\cdots dx_n, \tag{3.1}$$

and then

$$K_n = 2^n \theta^n \int_0^\infty \cdots \int_0^\infty \frac{\exp\left(-\theta({x_1}^2 + \cdots + {x_n}^2)\right)}{{x_1}^2 + \cdots + {x_n}^2} x_1 \cdots x_n dx_1 \cdots dx_n. \tag{3.2}$$

Now, we use the spherical coordinates, and set as in (2.1)

$$\begin{aligned}
x_1 &= r \sin\phi_{n-1} \cdots \sin\phi_3 \sin\phi_2 \sin\phi_1 \\
x_2 &= r \sin\phi_{n-1} \cdots \sin\phi_3 \sin\phi_2 \cos\phi_1 \\
x_3 &= r \sin\phi_{n-1} \cdots \sin\phi_3 \cos\phi_2 \\
\cdots &= \cdots\cdots\cdots\cdots \\
x_{n-1} &= r \sin\phi_{n-1} \cos\phi_{n-2} \\
x_n &= r \cos\phi_{n-1},
\end{aligned} \tag{3.3}$$

where $0 \le r < \infty$ and $0 \le \phi_i \le \frac{\pi}{2}$, for all $i = 1, \ldots, n-1$. Then, from (3.2), (3.3), and Fubini's Theorem, we get

$$K_n = 2^n \theta^n \left[\int_0^\infty r^{2n-3} \exp\left(-\theta r^2\right) dr\right] \prod_{i=1}^{n-1} \left[\int_0^{\frac{\pi}{2}} \sin^{2i-1}(\phi_i) \cos(\phi_i)\, d\phi_i\right]. \tag{3.4}$$

The first integral in (3.4) is easily computed by setting $t := \theta r^2$. Indeed, with this change of variable, we have:

$$\int_0^\infty r^{2n-3} \exp\left(-\theta r^2\right) dr = \frac{1}{2\theta^{n-1}} \int_0^\infty t^{n-2} \exp\left(-t\right) dt = \frac{\Gamma(n-1)}{2\theta^{n-1}} = \frac{(n-2)!}{2\theta^{n-1}}. \tag{3.5}$$

The product in (3.4) is easy to obtain. Indeed, we have

$$\prod_{i=1}^{n-1} \left[\int_0^{\frac{\pi}{2}} \sin^{2i-1}(\phi_i) \cos(\phi_i)\, d\phi_i\right] = \prod_{i=1}^{n-1} \frac{1}{2i} = \frac{1}{2^{n-1}(n-1)!}. \tag{3.6}$$

From (3.4), (3.5), and (3.6), we conclude that

$$K_n = \frac{\theta}{n-1}. \tag{3.7}$$

We deduce that

$$I_n = -\frac{\theta}{n-1}, \tag{3.8}$$

and

$$J_n = \frac{\theta}{n-1}. \tag{3.9}$$

So we recapture (in (3.8) and (3.9)) the results already obtained by K. L. D. Gunawardena in [1] by using probabilistic tools.

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References

1. K. L. D. Gunawardena, "On the Evaluation of Multiple Integrals," *Missouri Journal of Mathematical Sciences,* 6 (1994), 29–33.

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