REPRESENTING INTEGERS IN THE BINARY NUMBER SYSTEM AS PERMANENTS OF CERTAIN MATRICES

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Abstract. The permanent of an m–by–n matrix A is the sum of all possible products of m elements from A with the property that the elements in each of the products lie on different lines of A. This scalar valued function of the matrix A occurs throughout the combinatorial literature in connection with various enumeration and extremal problems. In this note, we construct a $(0, 1)$–matrix with a prescribed permanent, 1, 2, $\ldots$, 2^{n-1}.

1. Introduction. Let $A = [a_{ij}]$ be an m–by–n matrix. The *permanent* of A is defined by

$$\mathrm{per}(A) = \sum a_{1i_1} a_{2i_2} \cdots a_{mi_m},$$

where the summation extends over all the m–permutations $(i_1, i_2, \ldots , i_m)$ of the integers 1, 2, $\ldots$, n. Thus, $\mathrm{per}(A)$ is the sum of all possible products of m elements of the m–by–n matrix A with the property that the elements in the product comes from different columns of A. This scalar valued function of the matrix A occurs throughout the combinatorial literature in connection with various enumeration and extremal problems. In [1], it was shown that the existence of the $(0, 1)$–matrix of order n whose permanent is k $(0 \le k \le 2^{n-1})$. In this note, we construct a $(0, 1)$–matrix whose permanent is k $(0 \le k \le 2^{n-1})$ by the binary number system of k. Its related matrices are also considered.

2. Construction of $(\mathbf{0}, \mathbf{1})$–Matrix with Permanent k. For the purpose of our construction, we define the following matrix.

Definition 1. Let B_n be the following n–by–n $(0,1)$–matrix.

$$B_n = [b_{ij}] = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We say B_n is the *basic matrix* of order n.

Let $A = [a_{ij}]$ be an m–by–n real matrix with row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say A is *contractible* on column (row) k if column (row) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1)$–by–$(n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called the *contraction* of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0$ and $a_{kj} \neq 0$ for some $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A on row k relative to columns i and j. Now we can evaluate the permanent of B_n.

Lemma 1. $\mathrm{per}B_n = 2^{n-1}$.

Proof. By the contraction of B_n on column n relative to rows 1 and n, $\mathrm{per}B_n = 2\mathrm{per}B_{n-1}$. Since $\mathrm{per}B_2 = 2$, it follows by induction that $\mathrm{per}B_n = 2^{n-1}$. The proof is complete.

In addition, we may define a related matrix using B_n.

Definition 2. $B_n(n,k) = B_n - E_{n,k}$, where $E_{n,k}$ is a cell which is a $(0,1)$–matrix with only one entry equal to 1, the (n,k) entry.

Definition 3. $B_n(n|k)$ is the submatrix of B_n whose nth row and kth column are deleted.

Proposition. $\mathrm{per}B_n(n,k) = 2^{n-1} - 2^{k-2}$, $(k = 2,3,\ldots,n)$ and $\mathrm{per}B_n(n,1) = 2^{n-1} - 1$.

Proof. It easily follows that $\mathrm{per}B_n(n|k) = 2^{k-2}$, $k = 2, 3, \dots, n$, and $\mathrm{per}B_n(n|1) = 1$. Since $\mathrm{per}B_n(n,k) = \mathrm{per}B_n - \mathrm{per}B_n(n|k)$, $\mathrm{per}B_n(n,k) = 2^{n-1} - 2^{k-2}$, $k = 2, 3, \dots, n$ and $\mathrm{per}B_n(n,1) = 2^{n-1} - 1$. The proof is complete.

Now we define the process of the construction of a $(0,1)$–matrix by the binary number system (base 2), and give the main result.

Definition 4. Let t be a positive integer with $2^{n-2} \leq t \leq 2^{n-1}$ and $t_{(2)} = x_{n-2}x_{n-3} \cdots x_1 x_0$ be the binary representation of t, where the x_i's are 0 or 1. Define $x_k^c = 1$ if $x_k = 0$, and 0, otherwise.

Theorem. Let t be a positive integer with $2^{n-2} < t \leq 2^{n-1}$ and $t_{(2)} = x_{n-2}x_{n-3} \cdots x_1 x_0$ be the binary representation of t. Let $B_n(t)$ be obtained from B_n by replacing $b_{n,k}$ to x_{k-1}^c, for all $k = 1, 2, \dots, n-1$. Then $\mathrm{per}B_n(t) = 2^{n-1} - t$.

Proof. Let $t_{(2)} = x_{n-2}x_{n-3} \cdots x_1 x_0$. Then $t = x_{n-2}2^{n-2} + x_{n-3}2^{n-3} + \cdots + x_1 + x_0$. By the expansion of the nth row of $B_n(t)$,

$$
\begin{aligned}
\mathrm{per}(B_n(t)) &= \sum_{k=1}^{n} b_{nk} \mathrm{per}B(n|k) \\
&= \mathrm{per}B(n|1) + \sum_{k=2}^{n} x_{k-2}^c \mathrm{per}B(n|k) + \mathrm{per}B(n|n) \\
&= 1 + \sum_{k=2}^{n} x_{k-2}^c 2^{k-2} + 2^{n-2} \\
&= 2^{n-1} - \sum_{k=2}^{n} x_{k-2} 2^{k-2} \\
&= 2^{n-1} - t.
\end{aligned}
$$

The proof is complete.

Example. Let $k = 389$. Then

$$B_{10} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

with permanent $512 = 2^9$. Take $t = 123$. Then $(123)_{(2)} = 01111011$. Then

$$B_{10}(123) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and $\text{per}B_{10}(123) = 512 - 123 = 389$.

Conclusion. For any positive integer k, we can find a $(0,1)$–matrix with permanent k, $(1, 2, \ldots, 2^{n-1})$.

3. Some Results for Related $\mathbf{B_n}$. Now we may consider the permanent of matrices by replacing 0 entries in B_n by 1. The following theorem contains properties of permanents for matrices which are related to B_n. Recall that matrix $E_{i,j}$ is a cell with appropriate size.

Theorem.

(1) $\text{per}(B_n + E_{2,k}) = 2^{n-k}\text{per}(B_k + E_{2,k})$, $(k = 3, 4, \ldots, n-1)$.

(2) $\text{per}(B_n + E_{2,n}) = 2\text{per}B_{n-1} + \text{per}(B_{n-1} + E_{2,3} + E_{3,4} + \cdots + E_{n-2,n-1})$.

(3) $\mathrm{per}(B_n + E_{2,3} + E_{3,4} + \cdots + E_{n-1,n}) = 2 \times 3^{n-2}$.
(4) $\mathrm{per}(B_n + E_{3,k}) = 2^{n-k+1}\mathrm{per}(B_{k-1} + E_{2,k-1})$, $(k = 4, 5, \ldots, n-1)$.
(5) $\mathrm{per}(B_n + E_{k,n}) = 2^{k-2}\mathrm{per}(B_{n-k+2} + E_{2,n-k+2})$, $(k = 3, 4, \ldots, n-1)$.

Proof.

(1) By contraction of $B_n + E_{2,k}$ on column n relative to rows of 1 and n, we obtain $\mathrm{per}(B_n + E_{2,k}) = 2\mathrm{per}(B_{n-1} + E_{2,k})$, where the order of $E_{2,k}$ is $n-1$. Now we apply this process to $B_{n-1} + E_{2,k}$ of order $n-1$. Then $\mathrm{per}(B_n + E_{2,k}) = 2^{n-k}\mathrm{per}(B_k + E_{2,k})$, $(k = 3, 4, \ldots, n-1)$.
(2) Expanding the second row of $(B_n + E_{2,n})$, we obtain $\mathrm{per}(B_n + E_{2,n}) = 2\mathrm{per}B_{n-1} + \mathrm{per}(B_{n-1} + E_{2,3} + E_{3,4} + \cdots + E_{n-2,n-1})$.
(3) Expand the second row of $\mathrm{per}(B_n + E_{2,3} + E_{3,4} + \cdots + E_{n-1,n}) = 3\mathrm{per}(B_{n-1} + E_{2,3} + E_{3,4} + \cdots + E_{n-2,n-1})$. Continuing this process, we obtain the matrix J_3 with permanent equal to 3!.
(4) Contracting $B_n + E_{3,k}$ on row 2 relative to columns 1 and 2, we have that $\mathrm{per}(B_n + E_{3,k}) = 2\mathrm{per}(B_{n-1} + E_{2,k-1})$. Now applying formula (1) to the matrix $B_{n-1} + E_{2,k-1}$, we have that $\mathrm{per}(B_{n-1} + E_{2,k-1}) = 2^{n-k-2}\mathrm{per}(B_{k-1} + E_{2,k-1})$.
(5) Contracting $B_n + E_{k,n}$ on row 2 relative to columns 1 and 2, we have that $\mathrm{per}(B_n + E_{k,n}) = 2\mathrm{per}(B_{n-1} + E_{k-1,n-1})$. Continuing this process, we obtain the matrix $B_{n-k+2} + E_{2,n-k+2}$. This leads us to the conclusion.

The proof is complete.

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Reference

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