NEW ALGORITHMS FOR THE DIVISION'S QUOTIENT

S. Mazouzi

Abstract. We shall propose in this paper some practical divisibility tests as well as genuine expressions for the division's quotient. The latter is only based on successive operations using either addition or substraction and multiplication by some appropriate integer weights.

1. Introduction. As everybody knows, of all traditional known operations, division is the most intricate and requires time, patience, and even material resources when implementing the algorithm in hardware. In order to quickly recognize whether a positive integer number is divisible by another some attempts have been made throughout the years to develop a basic skill relevant to the divisibility tests which may relieve human beings of performing long arithmetical operations.

Besides the inherited divisibility tests by numbers less than or equal to 11 (see [1, 2, 5]) the ancient Spanish provided their own divisibility criterion for the number 13 [4]. A few years ago, inspired by the latter criterion, T. Trotter gave in [4] easy tests for divisibility by the numbers 17 and 19. The reader can eventually find through the number theory literature a great deal of rough results dealing with divisibility tests by specific numbers which require a rigorous theoretical framework.

We shall introduce in the present work new methods of testing divisibility of a positive integer number by another, and whenever it is true, we propose genuine algorithms to find the corresponding division's quotient. Meanwhile, we generalize (and simplify) the A. Khane results [3] by giving in an elementary way the exact *multiplier* for each division. The main technique we shall use is based on the natural properties of the "unit function" $\mathbf{u}\colon \mathbb{N} \to \{0, 1, \ldots, 9\}$ defined by $\mathbf{u}(n) = n - 10\lfloor \frac{n}{10} \rfloor$ (it is the function that associates to any nonnegative integer number its unit's digit); for instance, $\mathbf{u}(5248) = 8$ and $\mathbf{u}(219) = 9$. As usual, the brackets $\lfloor\cdot\rfloor$ denote the floor function. Next, we define the function $U\colon \mathbb{N}\times\mathbb{N} \to \{0, 1, \ldots, 9\}$ by $U(m, n) = \mathbf{u}(\mathbf{u}(m)\mathbf{u}(n))$, for any nonnegative integers m and n. The restriction of U to the set: $\{0, 1, \ldots, 9\} \times \{0, 1, \ldots, 9\}$ generates the following table of multiplication:

×	0	**1**	2	**3**	4	5	6	**7**	8	**9**
0	0	0	0	0	0	0	0	0	0	0
1	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2	0	2	4	6	8	0	2	4	6	8
3	**0**	**3**	**6**	**9**	**2**	**5**	**8**	**1**	**4**	**7**
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	**0**	**7**	**4**	**1**	**8**	**5**	**2**	**9**	**6**	**3**
8	0	8	6	4	2	0	8	6	4	2
9	**0**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**

It seems clear from the above table that if $n \in \mathbb{N}$ is fixed in the set $\{1,3,7,9\}$, that is n is coprime to the base $\mathbf{A} = 10$, then the section function $U(\cdot,n) = \mathbf{u}((\cdot)n)\colon \{0,1,\ldots,9\} \to \{0,1,\ldots,9\}$, denoted $\mathbf{u}_n$, is a one-to-one and onto mapping. As a matter of fact one can easily check that

1) $\mathbf{u}_1(m) = \mathbf{u}(m) = m,$ for all $m \in \{0,1,\ldots,9\}$,
2) $\mathbf{u}_3(m) = 10\lfloor\frac{5m+6}{7}\rfloor - 7m,$ for all $m \in \{0,1,\ldots,9\}$,
3) $\mathbf{u}_7(m) = 10\lfloor\frac{m+2}{3}\rfloor - 3m,$ for all $m \in \{0,1,\ldots,9\}$,
4) $\mathbf{u}_9(m) = 10 - m - 10\lfloor\frac{10-m}{10}\rfloor,$ for all $m \in \{0,1,\ldots,9\}$,
5) $\mathbf{u}_a(m) + \mathbf{u}_{10-a}(m) = 10 - 10\lfloor\frac{10-m}{10}\rfloor,$ for all $m \in \{0,1,\ldots,9\}$, for all $a \in \{1,3,7,9\}$,
6) $\mathbf{u}_a^{-1}(m) = 10\lfloor\frac{(a-2)m+a-1}{a}\rfloor - am,$ for all $m \in \{1,\ldots,9\},$ $a = 3,7$,
7) $\mathbf{u}_9^{-1} = \mathbf{u}_9$.

2. Results Relevant to the Decimal Base. We start with the following results which give simple algorithms to test the divisibility of a positive integer number by another and systematically find the corresponding quotient. The principle consists of finding the digits of the quotient proceeding from the unit's digit to the left most digit.

Theorem 1 (division from right to left). Let $N_0, K \in \mathbb{N}$ such that $b = \mathbf{u}(N_0) \neq 0$ and $K = 10p + a$, for some $a \in \{1, 3, 7, 9\}$. We define the sequence $\{N_k\}_{k \geq 0} \subset \mathbb{N}$ by

$$N_{k+1} = \frac{N_k - K\mathbf{u}_a^{-1}(\mathbf{u}(N_k))}{10}, \quad k = 0, 1, \dots . \tag{1}$$

Then, N_0 is divisible by K if and only if N_k is divisible by K for each $k \in \mathbb{N}$. Moreover, the quotient $L = N_0/K$ is given by

$$L = \sum_{k=0}^{\bar{n}} \mathbf{u}_a^{-1}(\mathbf{u}(N_k)) \cdot 10^k,$$

where $\bar{n}$ is the least positive integer number n such that $N_{n+1} = 0$.

Proof. Write $N_0 = 10A + b$. Then

$$\begin{aligned} N_0 &= 10A + b - K\mathbf{u}_a^{-1}(b) + K\mathbf{u}_a^{-1}(b) \\ &= 10\left(A - \frac{K\mathbf{u}_a^{-1}(b) - b}{10}\right) + K\mathbf{u}_a^{-1}(b). \end{aligned}$$

Since 10 and $K\mathbf{u}_a^{-1}(b)$ are coprime and K divides N_0 then K must divide

$$A - \frac{K\mathbf{u}_a^{-1}(b) - b}{10} = \frac{N_0 - K\mathbf{u}_a^{-1}(b)}{10} = N_1$$

and by induction K divides each term N_k.

It follows from relation (1) that

$$N_{k+1} \leq \frac{N_k}{10}, \quad k = 0, 1, \dots$$

so that,

$$N_{k+1} \leq \frac{N_0}{10^{k+1}}, \quad k = 0, 1, \ldots$$

and thus, $\lim_{k\to\infty} N_k = 0$. Therefore, there is a positive integer n such that $N_k = 0$ for all $k \geq n$. Denote by $\bar{n}$ the least positive integer n such that $N_{n+1} = 0$.

To find the quotient we observe that

$$10^{k+1} N_{k+1} = 10^k N_k - 10^k K \mathbf{u}_a^{-1}(\mathbf{u}(N_k)), \quad k = 0, 1, \ldots .$$

Then,

$$\begin{aligned} \sum_{k=0}^{\bar{n}} 10^{k+1} N_{k+1} = \sum_{k=1}^{\bar{n}} 10^k N_k &= \sum_{k=0}^{\bar{n}} 10^k N_k - \sum_{k=0}^{\bar{n}} 10^k K \mathbf{u}_a^{-1}(\mathbf{u}(N_k)) \\ &= KL + \sum_{k=1}^{\bar{n}} 10^k N_k - K \sum_{k=0}^{\bar{n}} \mathbf{u}_a^{-1}(\mathbf{u}(N_k)) 10^k, \end{aligned}$$

from which we get

$$L = \sum_{k=0}^{\bar{n}} \mathbf{u}_a^{-1}(\mathbf{u}(N_k)) 10^k.$$

Example 2. Let $N_0 = 2111103,\ K = 389$, then we have

$$\begin{array}{rcccccccccl}
N_0 & = & 2 & 1 & 1 & 1 & 1 & 0 & \mathbf{3} & & \\
 & - & & & & 2 & 7 & 2 & 3 & & (= K\mathbf{u}_9^{-1}(\mathbf{3})) \\
 & - & - & - & - & - & - & - & - & & \\
N_1 & = & 2 & 1 & 0 & 8 & 3 & \mathbf{8} & \emptyset & & \\
 & - & & & & 7 & 7 & 8 & & & (= K\mathbf{u}_9^{-1}(\mathbf{8})) \\
 & - & - & - & - & - & - & - & & & \\
N_2 & = & 2 & 1 & 0 & 0 & \mathbf{6} & \emptyset & & & \\
 & - & & 1 & 5 & 5 & 6 & & & & (= K\mathbf{u}_9^{-1}(\mathbf{6})) \\
 & - & - & - & - & - & - & & & & \\
N_3 & = & 1 & 9 & 4 & \mathbf{5} & \emptyset & & & & \\
 & - & 1 & 9 & 4 & 5 & & & & & (= K\mathbf{u}_9^{-1}(\mathbf{5})) \\
 & - & - & - & - & - & & & & & \\
N_4 & = & 0 & & & & & & & &
\end{array}$$

Hence, $L = 10^3\mathbf{u}_9^{-1}(\mathbf{5}) + 10^2\mathbf{u}_9^{-1}(\mathbf{6}) + 10\mathbf{u}_9^{-1}(\mathbf{8}) + \mathbf{u}_9^{-1}(\mathbf{3}) = 5427$.

Theorem 3 (division from right to left). Let $N_0, K \in \mathbb{N}$ such that $b = \mathbf{u}(N_0) \neq 0$ and $K = 10p+a$, for some $a \in \{1,3,7,9\}$. Let $\{N_k\}_{k\geq 0} \subset \mathbb{N}$ be a sequence defined by

$$N_{k+1} = \frac{N_k + K\mathbf{u}_{10-a}^{-1}(\mathbf{u}(N_k))}{10} \quad k = 0, 1, \dots . \tag{2}$$

Then, N_0 is divisible by K if and only if N_k is divisible by K for each $k \in \mathbb{N}$. Besides, the quotient $L = N_0/K$ is given by

$$L = \{10 - \mathbf{u}_{10-a}^{-1}(b)\} + \sum_{k=1}^{\bar{n}'}\{9 - \mathbf{u}_{10-a}^{-1}(\mathbf{u}(N_k))\} \cdot 10^k,$$

where $\bar{n}'$ is the least positive integer n such that $N_{n+1} = K$.

Proof. We have

$$N_0 = 10A + b = 10A + b + K\mathbf{u}_{10-a}^{-1}(b) - K\mathbf{u}_{10-a}^{-1}(b)$$

$$= 10\left(A + \frac{K\mathbf{u}_{10-a}^{-1}(b) + b}{10}\right) - K\mathbf{u}_{10-a}^{-1}(b).$$

Now since the numbers 10 and $K\mathbf{u}_{10-a}^{-1}(b)$ are coprime and K divides N_0, then K must divide the integer number

$$A + \frac{K\mathbf{u}_{10-a}^{-1}(b) + b}{10} = \frac{N_0 + K\mathbf{u}_{10-a}^{-1}(b)}{10} = N_1$$

and by induction K divides each N_k.

We observe on the other hand that

$$N_{k+1} - K = \frac{N_k - K}{10} - \frac{K(9 - \mathbf{u}_{10-a}^{-1}(\mathbf{u}(N_k)))}{10}, \quad k = 0, 1, \ldots$$

so that,

$$0 \le N_{k+1} - K \le \frac{N_k - K}{10} \le \cdots \le \frac{N_0 - K}{10^{k+1}}, \quad k = 0, 1, \ldots .$$

Thus, there is a positive integer n such that $N_{k+1} = K$, for all $k \ge n$. We denote by $\bar{n}'$ the least positive integer n such that $N_{n+1} = K$.

We have from relation (2)

$$10^{k+1}N_{k+1} = 10^k N_k + 10^k K\mathbf{u}_{10-a}^{-1}(\mathbf{u}(N_k)), \quad k = 0, 1, \ldots$$

then,

$$\sum_{k=0}^{\bar{n}'} 10^{k+1} N_{k+1} = 10^{\bar{n}'+1} K + \sum_{k=1}^{\bar{n}'} 10^k N_k$$

$$= \sum_{k=0}^{\bar{n}'} 10^k N_k + K \sum_{k=0}^{\bar{n}'} \mathbf{u}_{10-a}^{-1}(\mathbf{u}(N_k)) \cdot 10^k$$

$$= KL + \sum_{k=1}^{\bar{n}'} 10^k N_k + K \sum_{k=0}^{\bar{n}'} \mathbf{u}_a^{-1}(\mathbf{u}(N_k)) \cdot 10^k,$$

from which we get

$$L = 10^{\bar{n}'+1} - \sum_{k=0}^{\bar{n}'} \mathbf{u}_{10-a}^{-1}(\mathbf{u}(N_k)) \cdot 10^k$$

$$= \{10 - \mathbf{u}_{10-a}^{-1}(b)\} + \sum_{k=1}^{\bar{n}'} \{9 - \mathbf{u}_{10-a}^{-1}(\mathbf{u}(N_k))\} \cdot 10^k.$$

Example 4. Let $N_0 = 2111103,\ K = 389$, then we have

$$
\begin{array}{ccccccccccl}
N_0 & = & 2 & 1 & 1 & 1 & 1 & 0 & \mathbf{3} & & \\
 & + & & & & 1 & 1 & 6 & 7 & & (= K\mathbf{u}_1^{-1}(\mathbf{3})) \\
 - & - & - & - & - & - & - & - & - & & \\
N_1 & = & 2 & 1 & 1 & 2 & 2 & \mathbf{7} & \emptyset & & \\
 & + & & & 2 & 7 & 2 & 3 & & & (= K\mathbf{u}_1^{-1}(\mathbf{7})) \\
 - & - & - & - & - & - & - & - & & & \\
N_2 & = & 2 & 1 & 3 & 9 & \mathbf{5} & \emptyset & & & \\
 & + & & 1 & 9 & 4 & 5 & & & & (= K\mathbf{u}_1^{-1}(\mathbf{5})) \\
 - & - & - & - & - & - & - & & & & \\
N_3 & = & 2 & 3 & 3 & \mathbf{4} & \emptyset & & & & \\
 & + & 1 & 5 & 5 & 6 & & & & & (= K\mathbf{u}_1^{-1}(\mathbf{4})) \\
 - & - & - & - & - & - & & & & & \\
N_4 & = & 3 & 8 & 9 & \emptyset & & & & & (= \quad K)
\end{array}
$$

so that,

$$
\begin{aligned}
L &= 10^3(9 - \mathbf{u}_1^{-1}(\mathbf{4})) + 10^2(9 - \mathbf{u}_1^{-1}(\mathbf{5})) + 10(9 - \mathbf{u}_1^{-1}(\mathbf{7})) + (10 - \mathbf{u}_1^{-1}(\mathbf{3})) \\
&= 5427.
\end{aligned}
$$

Theorem 5 (Division via a positive multiplier). Let $N_0,\ K \in \mathbb{N}$ such that $b = \mathbf{u}(N_0)$ and $K = 10p + a$, for some $a \in \{1, 3, 7, 9\}$. We define the sequence $\{\tilde{N}_k\}_{k \geq 0} \subset \mathbb{N}$ by

$$
\begin{cases}
\tilde{N}_0 = N_0 \mathbf{u}_a^{-1}(9), \\
\tilde{N}_{k+1} = \dfrac{\tilde{N}_k + K\mathbf{u}_a^{-1}(9)\mathbf{u}(\tilde{N}_k)}{10}, \quad k = 0, 1, \ldots .
\end{cases}
\tag{3}
$$

Then, N_0 is divisible by K if and only if $\tilde{N}_k$ is divisible by K for each $k \in \mathbb{N}$. Furthermore, the quotient $L = N_0/K$ is given by

$$
L = \{10 - \mathbf{u}(b\mathbf{u}_a^{-1}(9))\} + \sum_{k=1}^{\bar{n}} \{9 - (\mathbf{u}(\tilde{N}_k))\} \cdot 10^k,
$$

where $\bar{n}$ is the least positive integer n such that $\tilde{N}_{n+1} = K\mathbf{u}_a^{-1}(9)$.

Proof. It is easy to show that the sequence $\{\tilde{N}_k\}_{k\geq 0}$ is nonincreasing and there is a positive integer n such that $\tilde{N}_{k+1} = K\mathbf{u}_a^{-1}(9)$, for all $k \geq n$. Let $\bar{n}$ be the least positive integer such that $\tilde{N}_{\bar{n}+1} = K\mathbf{u}_a^{-1}(9)$.

The first assertion of the theorem is similar to that of the preceding theorems and so it is omitted. It is worthwhile to observe that

$$\begin{aligned}\tilde{N}_{k+1} &= \frac{\tilde{N}_k + (K\mathbf{u}_a^{-1}(9))\mathbf{u}(\tilde{N}_k)}{10}\\ &= \frac{\tilde{N}_k - \mathbf{u}(\tilde{N}_k)}{10} + \frac{(K\mathbf{u}_a^{-1}(9)+1)}{10}\mathbf{u}(\tilde{N}_k), \quad k = 0, 1, \dots .\end{aligned}$$

We shall denote by $\mu^+(K)$ the positive integer $\dfrac{K\mathbf{u}_a^{-1}(9)+1}{10}$ called a *positive multiplier*. We have for different expressions of K the following values of the corresponding multipliers:

$$\mu^+(K) = \begin{cases} 9p+1, & \text{if } \mathbf{u}(K) = 1\\ 3p+1, & \text{if } \mathbf{u}(K) = 3\\ 7p+5, & \text{if } \mathbf{u}(K) = 7\\ p+1, & \text{if } \mathbf{u}(K) = 9. \end{cases}$$

To obtain explicitly the quotient $L = N_0/K$ we have from relation (3)

$$10^{k+1}\tilde{N}_{k+1} = 10^k\tilde{N}_k + 10^k K\mathbf{u}_a^{-1}(9)\mathbf{u}(\tilde{N}_k), \quad k = 0, 1, \dots$$

so that,

$$\sum_{k=0}^{\bar{n}} 10^{k+1}\tilde{N}_{k+1} = 10^{\bar{n}+1}K\mathbf{u}_a^{-1}(9) + \sum_{k=1}^{\bar{n}} 10^k\tilde{N}_k$$

$$= \sum_{k=0}^{\bar{n}} 10^k\tilde{N}_k + K\mathbf{u}_a^{-1}(9)\sum_{k=0}^{\bar{n}} \mathbf{u}(\tilde{N}_k)\cdot 10^k$$

$$= KL\mathbf{u}_a^{-1}(9) + \sum_{k=1}^{\bar{n}} 10^k\tilde{N}_k + K\mathbf{u}_a^{-1}(9)\sum_{k=0}^{\bar{n}} \mathbf{u}(\tilde{N}_k)\cdot 10^k,$$

which implies that

$$L = 10^{\bar{n}+1} - \sum_{k=0}^{\bar{n}} \mathbf{u}(\tilde{N}_k)\cdot 10^k$$

$$= \{10 - \mathbf{u}(b\mathbf{u}_a^{-1}(9))\} + \sum_{k=1}^{\bar{n}}\{9 - \mathbf{u}(\tilde{N}_k)\}\cdot 10^k,$$

as expected.

Example 6. Let $N_0 = 4779411$ and $K = 7503$, then $\tilde{N}_0 = N_0\mathbf{u}_3^{-1}(9) = 14338233$ and $K\mathbf{u}_3^{-1}(9) = 22509$. Hence, $\mu^+(K) = (K\mathbf{u}_3^{-1}(9)+1)/10 = 2251$, then

$$\begin{array}{cccccccccccl}
\tilde{N}_0 & = & 1 & 4 & 3 & 3 & 8 & 2 & 3 & \mathbf{3} & & \\
 & + & & & & 6 & 7 & 5 & 3 & & & \big(=\mu^+(K)\mathbf{u}(\tilde{N}_0)\big) \\
 & - & - & - & - & - & - & - & - & & & \\
\tilde{N}_1 & = & 1 & 4 & 4 & 0 & 5 & 7 & \mathbf{6} & & & \\
 & + & & 1 & 3 & 5 & 0 & 6 & & & & \big(=\mu^+(K)\mathbf{u}(\tilde{N}_1)\big) \\
 & - & - & - & - & - & - & - & & & & \\
\tilde{N}_2 & = & 1 & 5 & 7 & 5 & 6 & \mathbf{3} & & & & \\
 & + & & 6 & 7 & 5 & 3 & & & & & \big(=\mu^+(K)\mathbf{u}(\tilde{N}_2)\big) \\
 & - & - & - & - & - & - & & & & & \\
\tilde{N}_3 & = & 2 & 2 & 5 & 0 & 9 & & & & & (= \quad K\mathbf{u}_3^{-1}(9)
\end{array}$$

Since $\tilde{N}_3 = K\mathbf{u}_3^{-1}(9)$, then

$$L = 10^2(9-\mathbf{u}(\tilde{N}_2)) + 10^1(9-\mathbf{u}(\tilde{N}_1)) + (10-\mathbf{u}(\tilde{N}_0)) = 637.$$

Theorem 7 (Division via a negative multiplier). Let N_0, $K \in \mathbb{N}$ such that $b = \mathbf{u}(N_0)$ and $K = 10p+a$, for $a \in \{1,3,7,9\}$. Let $\{\tilde{N}_k\}_{k\geq 0} \subset \mathbb{N}$ be a sequence defined by

$$\begin{cases} \tilde{N}_0 = N_0\mathbf{u}_{10-a}^{-1}(9), \\ \tilde{N}_{k+1} = \dfrac{\tilde{N}_k - K\mathbf{u}_{10-a}^{-1}(9)\mathbf{u}(\tilde{N}_k)}{10}, \quad k = 0,1,\dots. \end{cases} \tag{4}$$

Then, N_0 is divisible by K if and only if $\tilde{N}_k$ is divisible by K for each $k \in \mathbb{N}$. Besides, the quotient $L = N_0/K$ is given by

$$L = \sum_{k=0}^{\bar{n}'} \mathbf{u}(\tilde{N}_k)\cdot 10^k,$$

where $\bar{n}'$ is the least positive integer n such that $\tilde{N}_{n+1} = 0$.

Proof. It can easily be proved that the sequence $\{\tilde{N}_k\}_{k\geq 0}$ is nonincreasing and there is a positive integer n such that $\tilde{N}_{k+1} = 0$, for all $k \geq n$. Let $\bar{n}$ be the least positive integer such that $\tilde{N}_{\bar{n}+1} = 0$.

As above the first assertion of the theorem is similar to that of the foregoing theorems. Next, we note that

$$\begin{aligned}\tilde{N}_{k+1} &= \frac{\tilde{N}_k - K\mathbf{u}_{10-a}^{-1}(9)\mathbf{u}(\tilde{N}_k)}{10} \\ &= \frac{\tilde{N}_k - \mathbf{u}(\tilde{N}_k)}{10} - \frac{(K\mathbf{u}_{10-a}^{-1}(9) - 1)}{10}\mathbf{u}(\tilde{N}_k), \quad k = 0, 1, \ldots .\end{aligned}$$

We shall designate by $\mu^-(K)$ the negative integer number $-\dfrac{K\mathbf{u}_{10-a}^{-1}(9) - 1}{10}$ called a *negative multiplier.* Here are the values of the negative multipliers for different expressions of K:

$$-\mu^-(K) = \begin{cases} p, & \text{if } \mathbf{u}(K) = 1 \\ 7p + 2, & \text{if } \mathbf{u}(K) = 3 \\ 3p + 2, & \text{if } \mathbf{u}(K) = 7 \\ 9p + 8, & \text{if } \mathbf{u}(K) = 9. \end{cases}$$

Let us now find the expression of the corresponding quotient L. Relation (4) gives

$$10^{k+1}\tilde{N}_{k+1} = 10^k\tilde{N}_k - 10^k K\mathbf{u}_{10-a}^{-1}(9)\mathbf{u}(\tilde{N}_k), \quad k = 0, 1, \ldots$$

so that,

$$\begin{aligned}\sum_{k=0}^{\bar{n}'} 10^{k+1}\tilde{N}_{k+1} &= \sum_{k=1}^{\bar{n}'} 10^{k}\tilde{N}_{k}\\ &= \sum_{k=0}^{\bar{n}'} 10^{k}\tilde{N}_{k} - K\mathbf{u}_{10-a}^{-1}(9)\sum_{k=0}^{\bar{n}'}\mathbf{u}(\tilde{N}_k)\cdot 10^k\\ &= KL\mathbf{u}_{10-a}^{-1}(9) + \sum_{k=1}^{\bar{n}'} 10^{k}\tilde{N}_{k} - K\mathbf{u}_{10-a}^{-1}(9)\sum_{k=0}^{\bar{n}'}\mathbf{u}(\tilde{N}_k)\cdot 10^k,\end{aligned}$$

from which we deduce that

$$L = \sum_{k=0}^{\bar{n}'}\mathbf{u}(\tilde{N}_k)\cdot 10^k.$$

Example 8. Let $N_0 = 4779411$ and $K = 7503$, then $\tilde{N}_0 = N_0\mathbf{u}_{10-3}^{-1}(9) = 33455877$ and $K\mathbf{u}_{10-3}^{-1}(9) = 52521$. Hence, $\mu^-(K) = -(K\mathbf{u}_7^{-1}(9) - 1)/10 = -5252$, then

$$
\begin{array}{cccccccccccl}
\tilde{N}_0 & = & 3 & 3 & 4 & 5 & 5 & 8 & 7 & \mathbf{7} & & \\
 & - & & & 3 & 6 & 7 & 6 & 4 & & & \big(= \mu^-(K)\mathbf{u}(\tilde{N}_0)\big) \\
 & - & - & - & - & - & - & - & - & & & \\
\tilde{N}_1 & = & 3 & 3 & 0 & 8 & 8 & 2 & \mathbf{3} & & & \\
 & - & & 1 & 5 & 7 & 5 & 6 & & & & \big(= \mu^-(K)\mathbf{u}(\tilde{N}_1)\big) \\
 & - & - & - & - & - & - & - & & & & \\
\tilde{N}_2 & = & 3 & 1 & 5 & 1 & 2 & \mathbf{6} & & & & \\
 & - & & 3 & 1 & 5 & 1 & 2 & & & & \big(= \mu^-(K)\mathbf{u}(\tilde{N}_2)\big) \\
 & - & - & - & - & - & - & & & & & \\
\tilde{N}_3 & = & 0 & & & & & & & & &
\end{array}
$$

Hence,

$$L = 10^2\mathbf{u}(\tilde{N}_2) + 10^1\mathbf{u}(\tilde{N}_1) + \mathbf{u}(\tilde{N}_0) = 637.$$

Remark 1. Since

$$\mu^-(K) = -\frac{K\mathbf{u}_{10-a}^{-1}(9) - 1}{10} = -\frac{K(10 - \mathbf{u}_a^{-1}(9)) - 1}{10} = -K + \frac{K\mathbf{u}_a^{-1}(9) + 1}{10},$$

then,

$$\mu^+(K) - \mu^-(K) = K.$$

Proposition 9. A positive integer $N = \sum_{k=0}^m b_k 10^k$ is divisible by $K = 10p + a$, for some $a \in \{1, 3, 7, 9\}$ if and only if the positive integer $N^* = \left|\sum_{k=0}^m b_k \mu^{m-k}\right|$ is divisible by K, where μ is either $\mu^+(K)$ or $\mu^-(K)$.

Proof. We have for the first case $\mu = \mu^+(K) = \dfrac{K\mathbf{u}_a^{-1}(9)+1}{10}$ from which we get $10 = \dfrac{K\mathbf{u}_a^{-1}(9)+1}{\mu}$, therefore,

$$N = \sum_{k=0}^{m} b_k 10^k = \sum_{k=0}^{m} b_k \left(\frac{K\mathbf{u}_a^{-1}(9)+1}{\mu}\right)^k$$

$$= b_0 + \sum_{k=1}^{m} b_k \left\{1 + K\sum_{l=1}^{k} C_k^l K^{l-1}(\mathbf{u}_a^{-1}(9))^l\right\}\mu^{-k}$$

$$= \sum_{k=0}^{m} b_k\mu^{-k} + K\sum_{k=0}^{m}\left\{\sum_{l=1}^{k} C_k^l K^{l-1}(\mathbf{u}_a^{-1}(9))^l\right\}\mu^{-k}.$$

Multiplying both sides by μ^m we get

$$\sum_{k=0}^{m} b_k\mu^{m-k} = N\mu^m - K\sum_{k=0}^{m}\left\{\sum_{l=1}^{k} C_k^l K^{l-1}(\mathbf{u}_a^{-1}(9))^l\right\}\mu^{m-k}.$$

Since the right hand side is divisible by K, then $\sum_{k=0}^{m} b_k\mu^{m-k}$ is also divisible by K and conversely.

The second case can be dealt with in the same manner by merely observing that $10 = \dfrac{1 - K\mathbf{u}_a^{-1}(9)}{\mu}$.

Theorem 10 (Division by complementation). Let N_0 and K be two positive integer numbers such that $\mathbf{u}(N_0) \neq 0$ and $K = 10p + a$, with $a \in \{1, 3, 7, 9\}$.

We define the sequence $\{\bar{N}_k\}_{k\geq 0} \subset \mathbb{N}$ by

$$\begin{cases} \bar{N}_0 = 10^m - N_0\mathbf{u}_a^{-1}(9), & \\ \bar{N}_{k+1} = \dfrac{\bar{N}_k + K\mathbf{u}_a^{-1}(9)\mathbf{u}(\bar{N}_k)}{10}, & k \geq 0, \\ m \text{ being the number of digits of } N_0\mathbf{u}_a^{-1}(9). & \end{cases} \tag{5}$$

If N_0 is divisible by K then the quotient L is given by

$$L = \sum_{k=0}^{n-1} \mathbf{u}(\bar{N}_k)10^k,$$

where n is the least positive integer such that $10^n\bar{N}_n = 10^m$.

Proof. It can be proved that

$$\bar{N}_k = 10^{m-k} - \tilde{N}_k + K\mathbf{u}_a^{-1}(9), \quad k = 1, 2, \ldots$$

where $\{\tilde{N}_k\}_{k\geq 0}$ is defined by (3).

Let n be the least positive integer satisfying $\tilde{N}_n - K\mathbf{u}_a^{-1}(9) = 0$, then $\bar{N}_n = 10^{m-n}$, or equivalently, $10^n\bar{N}_n = 10^m$.

Relation (5) yields

$$\sum_{k=0}^{n-1} \bar{N}_{k+1}10^{k+1} = \sum_{k=0}^{n-1} \bar{N}_k 10^k + K\mathbf{u}_a^{-1}(9)\sum_{k=0}^{n-1} \mathbf{u}(\bar{N}_k)10^k,$$

so that,

$$\bar{N}_n 10^n \equiv 10^m = \bar{N}_0 + K\mathbf{u}_a^{-1}(9)\sum_{k=0}^{n-1} \mathbf{u}(\bar{N}_k)10^k,$$

hence,

$$N_0\mathbf{u}_a^{-1}(9) = 10^m - \bar{N}_0 = K\mathbf{u}_a^{-1}(9)\sum_{k=0}^{n-1}\mathbf{u}(\bar{N}_k)10^k.$$

Therefore,

$$L = \sum_{k=0}^{n-1}\mathbf{u}(\bar{N}_k)10^k.$$

Example 11. Let $N_0 = 520429$, $K = 637$, then $N_0\mathbf{u}_7^{-1}(9) = 3643003$ and $\hat{K} = K\mathbf{u}_7^{-1}(9) = 4459$, so that $m = 7$ and thus, $\bar{N}_0 = 6356997$.

$$\begin{array}{ccccccccccl}
\bar{N}_0 & = & & 6 & 3 & 5 & 6 & 9 & 9 & \mathbf{7} & \\
 & + & & & & 3 & 1 & 2 & 1 & 3 & \big(= \hat{K}\mathbf{u}(\bar{N}_0)\big) \\
 & - & - & - & - & - & - & - & - & - & \\
\bar{N}_1 & = & & 6 & 3 & 8 & 8 & 2 & \mathbf{1} & \emptyset & \\
 & + & & & & 4 & 4 & 5 & 9 & & \big(= \hat{K}\mathbf{u}(\bar{N}_1)\big) \\
 & - & - & - & - & - & - & - & - & & \\
\bar{N}_2 & = & & 6 & 4 & 3 & 2 & \mathbf{8} & \emptyset & & \\
 & + & & 3 & 5 & 6 & 7 & 2 & & & \big(= \hat{K}\mathbf{u}(\bar{N}_2)\big) \\
 & - & - & - & - & - & - & - & & & \\
\bar{N}_3 & = & 1 & 0 & 0 & 0 & 0 & \emptyset & & &
\end{array}$$

Since $10^3\bar{N}_3 = 10^m$, $(m = 7)$, then the quotient N_0 over K is $L = 817$.

3. Generalization to Other Bases. It is sometimes practical to handle some special forms of the integer if we want to test its divisibility by another integer, rather than using its decimal representation. For this reason we shall obtain similar divisibility criteria to those we have already obtained in the decimal base. We have the following results:

Theorem 12. Let $\mathbf{A}$ and K be two coprime positive integers nonequal to 0 and 1. A positive integer $N_0 = \mathbf{A}m + b$, with $0 \leq b < \mathbf{A}$, is divisible by K if and only if

$$N_1 = m + \frac{\delta K + 1}{\mathbf{A}} b$$

is divisible by K. (δ being the least positive integer such that $\dfrac{\delta K + 1}{\mathbf{A}}$ is a positive integer).

Proof. The fact that $\mathbf{A}$ and K are coprime implies the existence of two (nonunique) positive integers δ and μ such that $\mu\mathbf{A} - \delta K = 1$. Then,

$$N_0 = \mathbf{A}m + b = \mathbf{A}m + b + \delta bK - \delta bK$$

$$= \mathbf{A}\left(m + \frac{\delta K + 1}{\mathbf{A}} b\right) - \delta bK.$$

As K is coprime to $\mathbf{A}$ and divides N_0 then it must divide $N_1 = m + \dfrac{\delta K + 1}{\mathbf{A}} b$.

Theorem 13. Let $\mathbf{A}$ and K be two coprime positive integers nonequal to 0 and 1. A positive integer $N_0 = \mathbf{A}m + b$, with $0 \leq b < \mathbf{A}$, is divisible by K if and only if

$$N_1 = m - \frac{\delta' K - 1}{\mathbf{A}} b$$

is divisible by K. (δ' being the least positive integer such that $\dfrac{\delta' K - 1}{\mathbf{A}}$ is a positive integer).

Proof. Since $\mathbf{A}$ and K are coprime then there are two (nonunique) positive integers δ' and μ' such that $\delta' K - \mu' \mathbf{A} = 1$. It follows that

$$N_0 = \mathbf{A}m + b = \mathbf{A}m + b + \delta' bK - \delta' bK$$

$$= \mathbf{A}\left(m - \frac{\delta' K - 1}{\mathbf{A}} b\right) + \delta' bK.$$

As K is coprime to $\mathbf{A}$ and divides N_0 then it must divide $N_1 = m - \dfrac{\delta' K - 1}{\mathbf{A}} b$.

Remark 2. If we define $r_{\mathbf{A}}(S)$ to be the remainder of the euclidean division of S by $\mathbf{A}$, then K divides N_0 if and only if K divides each term of the sequence $\{N_k\}_{k\geq 0} \subset \mathbb{N}$ defined by

$$N_{k+1} = \frac{N_k + \delta K r_{\mathbf{A}}(N_k)}{\mathbf{A}}, \quad k = 0, 1, \dots .$$

Theorem 14. Let $\mathbf{A}$ and K be two coprime positive integers nonequal to 0 and 1. A positive integer $N = \sum_{k=0}^{n} b_k \mathbf{A}^k$ is divisible by K if and only if the positive integer $N^* = \left| \sum_{k=0}^{n} b_k \mu^{n-k} \right|$ is divisible by K (μ is either defined by $\mu = (\delta K + 1)/\mathbf{A}$ or $\mu = -(\delta' K - 1)/\mathbf{A}$).

Proof. Let us only consider the first case, that is $\mu = (\delta K + 1)/\mathbf{A}$, then $\mathbf{A} = (\delta K + 1)/\mu$. It follows that

$$\begin{aligned} N &= \sum_{k=0}^{n} b_k \left(\frac{\delta K + 1}{\mu} \right)^k = \sum_{k=0}^{n} b_k \left\{ \sum_{l=0}^{k} \mathbf{C}_k^l (\delta K)^l \right\} \mu^{-k} \\ &= \sum_{k=0}^{n} b_k \left\{ 1 + K \sum_{l=1}^{k} \mathbf{C}_k^l \delta^l K^{l-1} \right\} \mu^{-k}, \end{aligned}$$

hence,

$$\sum_{k=0}^{n} b_k \mu^{n-k} = N \mu^n - K \sum_{k=0}^{n} b_k \left\{ \sum_{l=1}^{k} \mathbf{C}_k^l \delta^l K^{l-1} \right\} \mu^{n-k}.$$

Therefore, N is divisible by K if and only if $\sum_{k=0}^{n} b_k \mu^{n-k}$ is divisible by K.

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S. Mazouzi
Department of Mathematics
Faculty of Sciences
University of Annaba
P. O. Box 12
Annaba 23000
Algeria
email: mazouzi_sa@yahoo.fr