# RETURNS TO THE ORIGIN FOR RANDOM WALKS ON $\mathbb{Z}$ REVISITED

Helmut Prodinger

Chrysafi and Bradley [1] considered symmetric random walks, defined as follows: let $X_k, k = 1, 2, \dots$ be independent and identically distributed random variables with $\mathbb{P}\{X_k = 1\} = \mathbb{P}\{X_k = -1\} = \frac{1}{2}$. Then

$$S_m = \sum_{k=1}^{m} X_k \quad \text{with} \quad S_0 = 0$$

is a simple random walk starting at 0. The authors considered only walks of even length $m = 2n$ and were interested in the random variable $R = R_n$, defined to be the *number of returns to the origin* in a walk of length $2n$, i.e., the number of times $S_i = 0$ happens, for $i = 1, \dots, 2n$. They computed moments up to $\mathbb{E}[R^6]$ and asked for a closed formula for $\mathbb{E}[R^k]$ and also whether $\mathbb{E}[R^k] \sim c_k n^{k/2}$ holds.

The answers to these questions can be found in [4] as opposed to [1]. There, the *factorial moments* $\mathbb{E}[R^{\underline{k}}]$ were computed. We state the formula only for even $n$:

$$\mathbb{E}[R_n^{\underline{k}}] = k! \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{\frac{i}{2} + n}{n}.$$

Ordinary moments can be recovered from these formulae as linear combinations with *Stirling numbers of the second kind (Stirling subset numbers),* see [3].

$$\mathbb{E}[R_n^k] = \sum_{i=0}^{k} \left\{ {k \atop i} \right\} \mathbb{E}[R_n^{\underline{i}}].$$

To answer the asymptotics question, we use generating functions

$$\mathbb{E}[R_n^{\underline{k}}] = 4^{-n} k! [z^n] \frac{(1 - \sqrt{1 - 4z})^k}{(1 - 4z)^{1 + \frac{k}{2}}},$$

and *singularity analysis of generating functions,* as nicely described in [2]. One must expand around the dominant singularity (here $z = \frac{1}{4}$),

$$\frac{(1-\sqrt{1-4z})^k}{(1-4z)^{1+\frac{k}{2}}} \sim (1-4z)^{-1-\frac{k}{2}},$$

and use a transfer theorem

$$\mathbb{E}[R_n^k] \sim 4^{-n} k! [z^n](1-4z)^{-1-\frac{k}{2}} = k! \binom{\frac{k}{2}+n}{n} \sim \frac{k!}{(k/2)!} n^{k/2}.$$

For odd $k = 2j+1$, the factor may be rewritten as follows:

$$\frac{k!}{(k/2)!} = \frac{(2j+1)!}{(j+\frac{1}{2})!} = \frac{2^{2j+1} j!}{\sqrt{\pi}}.$$

For ordinary moments, the leading terms in the asymptotic expansion are the same.

$$\mathbb{E}[R_n^k] \sim \frac{k!}{(k/2)!} n^{k/2}.$$

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*References*

1. L. A. Chrysafi and R. E. Bradley, "Returns to the Origin for Random Walks on $\mathbb{Z}$," *Missouri Journal of Mathematical Sciences,* 14 (2002), 96–106.

2. P. Flajolet and A. Odlyzko, "Singularity Analysis of Generating Functions," *SIAM Journal on Discrete Mathematics,* 3 (1990), 216–240.

3. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science,* Addison–Wesley Publishing Company, Amsterdam, 2nd edition, 1994.

4. P. Kirschenhofer and H. Prodinger, “The Higher Moments of the Number of Returns of a Simple Random Walk,” *Adv. in Appl. Probab.,* 26 (1994), 561–563.
5. P. Kirschenhofer and H. Prodinger, “Return Statistics of Simple Random Walks,” *J. Statist. Plann. Inference,* 54 (1996), 67–74.

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Helmut Prodinger
Mathematics Department
Stellenbosch University
7602 Stellenbosch, South Africa
email: hproding@sun.ac.za