A COMMON GENERALIZATION OF THE INTERMEDIATE VALUE THEOREM AND ROUCHÉ'S THEOREM

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Abstract. A simple proof of a theorem unifying Bolzano's Theorem [8], the Intermediate Value Theorem, Rouché's Theorem [3] and its extensions to differentiable maps to $\mathbb{R}^n$ [2, 6, 9] is obtained. This unifying theorem in particular shows that in Professor Baker's [1] examples where the number of solutions of $f(x) = y$ for a continuous map $f\colon B^2 \to \mathbb{R}^2$, $y \notin f(\partial B^2)$, from the unit ball B^2 in the plane $\mathbb{R}^2$ is not exactly the absolute value of the winding number of the curve $f(\partial B^2)$ about y, the number of the connected components of the solution set counted with multiplicity coincides with the winding number.

1. Introduction. Professor Shih [8] has observed that Bolzano's Theorem, an equivalent of the Intermediate Value Theorem, may be stated as follows: If f is a real-valued continuous function on the closed interval $I = [-1, 1]$ and $xf(x) > 0$ for $x \in \partial I$, the boundary of I, then f has at least one zero in I. Theorem A presents an analogue of Bolzano's Theorem [8] and Rouché's Theorem [1, 3] for the n-dimensional complex plane $\mathbb{C}^n$.

<u>Theorem A</u>. Let Ω be a bounded domain in $\mathbb{C}^n$ and let $f, g\colon \overline{\Omega} \to \mathbb{C}^n$ be continuous and holomorphic in Ω. Then

1. [9] The map f has exactly one zero in Ω if the origin $O \in \Omega$ and $Re(\overline{z} \cdot f(z)) > 0$ for $z \in \partial\Omega$.
2. [2, 6] The maps f and g have the same number of zeros counting multiplicity if
$$|f(z) - g(z)| < |f(z)| \text{ for } z \in \partial\Omega.$$

In this note, using the notion of intersection number defined for continuous maps [7], a simple proof of a unifying generalization of Theorem A to continuous maps in higher dimensions is obtained. The proof is simple, direct, and accessible.

2. Preliminaries. Definitions and terminologies are adopted from Guillemin and Pollack [4]. Let $f\colon X \to Y$ be a smooth map from a manifold X with boundary ∂X to a manifold Y. Submanifolds and the boundary of a manifold with orientation are as usual provided with induced orientations. A point $y \in Y$ is a regular value of $f\colon X \to Y$ if $df_x(T_x(X)) = T_y(Y)$ for $x \in f^{-1}(y)$, where df_x denotes the differential map of $T_x(X)$, the tangent space of X at x. The set of regular values of f, $R(f)$, is dense in Y by

Sard's Theorem [4]. Suppose $X \subset \mathbb{R}^n$ is a domain and $A \subset \overline{X}$. The maps $f, g\colon \overline{X} \to Y$ are (smoothly in X) [mod (A, y)] homotopic if there is a continuous map $F\colon (\alpha, \beta) \times \overline{X} \to Y$, $[0,1] \subset (\alpha, \beta)$, (smooth in $(\alpha, \beta) \times X$), $[y \in Y - F([0,1] \times A)]$ with $F(0, z) = f(z)$ and $F(1, z) = g(z)$. If $f\colon X \to Y$ is a map and $D \subset X$, the restriction of f to D will be denoted by f_D.

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f\colon \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable in Ω with $y \in \mathbb{R}^n - f(\partial\Omega)$. The intersection number, $I(f, y)$, of $f(\Omega)$ with y is defined as $\limsup_{y_i \to y}\{N(f, y_i) : y_i \in R(f)\}$, where $N(f, y_i)$ denotes the total number of preimages $x \in f^{-1}(y_i)$ counting orientation, i.e., with a preimage x making a contribution $+1$ or -1 depending whether the determinant of $(df)_x$ is positive or negative, respectively.

Proposition 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f\colon \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable in Ω with $y \in \mathbb{R}^n - f(\partial\Omega)$. Then $I(f, w) = I(f, y)$ for $w \in W$, the component of $\mathbb{R}^n - f(\partial\Omega)$ containing y.

Proof. Since $W \cap R(f)$ is connected and dense in W, it suffices to show that the map h defined by $h(z) = N(f, z)$ is locally constant in $W \cap R(f)$. Let $z \in W \cap R(f)$. If $f^{-1}(z) = \emptyset$, then there is a neighborhood $V \subset W$ of z such that $f^{-1}(V) = \emptyset$ and thus, $h(w) = 0$ for $w \in V$. Suppose $f^{-1}(z) = \{x_1, \dots, x_k\}$. There is an open neighborhood $V \subset W$ of z and open neighborhoods U_i of x_i such that (i) $f^{-1}(V) = U_1 \cup \cdots \cup U_k$, (ii) the sets $\overline{U}_i \subset \Omega$ are pairwise disjoint, and (iii) each $f_i\colon U_i \to V$ is a diffeomorphism. This shows $h(w) = h(z)$ for $w \in V$.

Proposition 2. Let Ω be a bounded domain in $\mathbb{R}^n$ and $[0,1] \subset (\alpha, \beta)$. Let $G\colon (\alpha, \beta) \times \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable in $(\alpha, \beta) \times \Omega$ with $y \in \mathbb{R}^n - G((\alpha, \beta) \times \partial\Omega)$. Then $I(F_0, y) = I(F_1, y)$, where $F_t(x) = G(t, x)$.

Proof. Let $t_0 \in [0,1]$. From the compactness of $[0,1]$, it is enough to show that there is an $\epsilon > 0$ such that $I(F_t, y) = I(F_{t_0}, y)$ if $t \in (t_0 - \epsilon, t_0 + \epsilon)$. The map H on $(\alpha, \beta) \times \overline{\Omega}$ defined by $H(t, x) = (t, F_t(x))$ is differentiable and $JH(t, x) = \det \frac{\partial H(t,z)}{\partial(t,z)} = \det \frac{\partial F_t(x)}{\partial(x)}$ for $x \in \Omega$.

If $H^{-1}(t_0, y) = \emptyset$, there is a neighborhood V of (t_0, y) with (i) $H^{-1}(V) = \emptyset$ and (ii) $[t_0 - \epsilon, t_0 + \epsilon] \times B_r(y) \subset V$ for $\epsilon > 0$ and $r > 0$ where $B_r(y) = \{w \in \mathbb{R}^n : |w - y| < r\}$. Since for $t \in [t_0 - \epsilon, t_0 + \epsilon]$, $F_t^{-1}(B_r(y)) = \emptyset$, we get $I(F_t, y) = I(F_{t_0}, y) = 0$.

Suppose $H^{-1}(t_0, y) \neq \emptyset$. Let V be a neighborhood of (t_0, y) with (i) $H^{-1}(V) \subset (\alpha, \beta) \times \Omega$ and (ii) $[t_0 - \epsilon_1, t_0 + \epsilon_1] \times B_{r'}(y) \subset V$ for $\epsilon_1 > 0$ and $r' > 0$. Let $p \in B_{r'}(y) \cap R(F_{t_0})$. If $H^{-1}(t_0, p) = \emptyset$, we have $I(F_t, p) =$

$I(F_{t_0}, p) = 0$, as in the previous case, for $t \in [t_0 - \epsilon, t_0 + \epsilon]$ with $0 < \epsilon < \epsilon_1$ and hence, $I(F_t, y) = I(F_{t_0}, y) = 0$ from Proposition 1. Now suppose $H^{-1}(t_0, p) = \{z_1, \ldots, z_k\}$. Since $(t_0, p) \in R(H)$ and so $JH(t_0, z_i) \neq 0$, there are $0 < \epsilon' < \epsilon_1$ and open neighborhoods U_i of z_i such that (i) $\overline{U}_i$ are pairwise disjoint and (ii) H is diffeomorphic on $(t_0 - \epsilon', t_0 + \epsilon') \times U_i$. Then F_t is also diffeomorphic on U_i for $t \in (t_0 - \epsilon', t_0 + \epsilon')$. There are numbers $r > 0$ and $\epsilon > 0$ such that $(t_0 - \epsilon, t_0 + \epsilon) \times \overline{B_r(p)} \subset (t_0 - \epsilon', t_0 + \epsilon') \times B_{r'}(y) \subset \bigcap_i H((t_0 - \epsilon', t_0 + \epsilon') \times U_i)$. It follows that for $t \in (t_0 - \epsilon, t_0 + \epsilon)$, $F_t^{-1}(B_r(p)) \subset \bigcup_i U_i$ and so $I(F_t, p) = I(F_{t_0}, p)$ and thus from Proposition 1, $I(F_t, y) = I(F_{t_0}, y)$.

Definition 2. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $f\colon \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable in Ω with $y \in \mathbb{R}^n - f(\partial\Omega)$. Let $A \subset \Omega$ be a component of $f^{-1}(y)$. The order of multiplicity of f at A, $\mu_A(f)$, is defined by $\mu_A(f) = I(f_{\overline{U}}, y)$, where $U \subset \Omega$ is a domain containing A with $\overline{U} \cap f^{-1}(y) = A$.

Remark. This definition agrees with that of order of multiplicity of holomorphic maps in $\mathbb{C}^n$ at an isolated preimage point [2].

Theorem 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f\colon \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable in Ω with $y \in \mathbb{R}^n - f(\partial\Omega)$. Then

(1). $f^{-1}(y) \neq \emptyset$ if $I(f, y) \neq 0$.

(2). $I(f, y) = I(g, y)$ if $g\colon \overline{\Omega} \to \mathbb{R}^n$ is mod $(\partial\Omega, y)$ homotopic smoothly in Ω to f.

(3). $I(f, y) = \sum_i \mu_{A_i}(f)$, where $\{A_1, \ldots, A_k\}$ are the components of $f^{-1}(y)$.

Proof.

(1). Assume $I(f, y) \neq 0$. A sequence $y_k \in R(f)$ converges to y and a sequence $x_k \in f^{-1}(y_k)$ has a limit point $x \in \Omega$ with $f(x) = y$.

(2). This follows from Proposition 2.

(3). Let U_i be open domains such that (i) $A_i \subset U_i \subset \overline{U}_i \subset \Omega$, (ii) $\overline{U}_i$ are pairwise disjoint, and (iii) $y \notin f(\bigcup_i \partial U_i \cup (\partial\Omega))$. Let $y_k \in R(f)$ be a sequence converging to y such that $f^{-1}(y_k) \subset \bigcup U_i$. Now $I(f, y) = \limsup_k I(f, y_k) = \limsup_k \sum_i I(f_{\overline{U}_i}, y_k) = \sum_i I(f_{\overline{U}_i}, y) = \sum_i \mu_{A_i}(f)$.

3. Main Results. An extension of the definition of intersection number to continuous $f\colon \overline{\Omega} \to \mathbb{R}^n$ with $y \in \mathbb{R}^n - f(\partial\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ is now given. By the Stone-Weierstrass Theorem [5] there is a sequence of polynomials $P_j\colon \overline{\Omega} \to \mathbb{R}^n$ which converges uniformly to f. The following lemma will facilitate a proof of Theorem 4, extending the collection of maps in Theorem 3 to a collection of continuous maps.

Convergence Lemma. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $f:\overline{\Omega} \to \mathbb{R}^n$ be continuous with $y \in \mathbb{R}^n - f(\partial\Omega)$. Suppose P_k, Q_j are sequences of polynomials converging uniformly to f on $\overline{\Omega}$. Then $I(P_j, y) = I(Q_k, y)$ for $j,\ k \geq N$ for a number N.

Proof. Let $r = \inf_{z\in\partial\Omega}\{|y - z| > 0\}$. There is an integer N such that

$$\sup\{|g(x) - f(x)| : g(x) = P_k(x) \text{ or } g(x) = Q_j(x), x \in \partial\Omega, j, k > N\} < \frac{r}{4}.$$

Let $k, j > N$ and define $F_{k,j}: [0,1] \times \overline{\Omega} \to \mathbb{R}^n$ by $F_{k,j}(t, x) = tP_k(x) + (1-t)Q_j(x)$. Since $y \notin F_{k,j}([0,1] \times \partial\Omega)$, $F_{k,j}$ defines a smoothly in Ω mod $(\partial\Omega, y)$ homotopy of the polynomials $P_k, Q_j: \overline{\Omega} \to \mathbb{R}^n$. The conclusion follows from Theorem 3(2).

By the Convergence Lemma the notions of intersection number and order of multiplicity may be extended to continuous maps.

Definition 3. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $f:\overline{\Omega} \to \mathbb{R}^n$ be a continuous map. Let $y \in \mathbb{R}^n - f(\partial\Omega)$ and let A be a component of $f^{-1}(y)$.

(1). The intersection number, $I(f, y)$, of $f(\Omega)$ with y is defined by $I(f, y) = \limsup I(P_i, y)$, where $P_i: \overline{\Omega} \to \mathbb{R}^n$ is a sequence of polynomials which converges uniformly to f.

(2). The order of multiplicity, $\mu_A(f)$, of f at A is defined by $\mu_A(f) = \limsup I(f_{\overline{U}}, y)$, where $U \subset \Omega$ is a domain containing A with $\overline{U} \cap f^{-1}(y) = A$.

Theorem 4 below, the main result, is a common generalization of Bolzano's Theorem, the Intermediate Value Theorem, Rouché's Theorem and Theorem A as Corollary 5 demonstrates.

Theorem 4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f:\overline{\Omega} \to \mathbb{R}^n$ be continuous with $y \in \mathbb{R}^n - f(\partial\Omega)$. Then

(1). $f^{-1}(y) \neq \emptyset$ if $I(f, y) \neq 0$.

(2). $I(f, y) = I(g, y)$ if $g:\overline{\Omega} \to \mathbb{R}^n$ is continuous and $\partial f, \partial g: \partial\Omega \to \mathbb{R}^n - \{y\}$ are homotopic.

(3). $I(f, y) = \sum_i \mu_{A_i}(f)$, where $\{A_1, \ldots, A_k\}$ are the components of $f^{-1}(y)$.

Proof.

(1). Suppose $P_j:\overline{\Omega} \to \mathbb{R}^n$ is a sequence of polynomials which converges uniformly to f such that $I(P_j, y) = I(f, y)$. Then $I(P_j, y) \neq 0$ for all j

and so for a sequence $x_j \in \Omega$, $P_j(x_j) = y$. Then $f(x) = y$ where $x \in \Omega$ is a limit point of a convergent subsequence of x_j.

(2). Let $G: ([0,1] \times \partial\Omega) \to \mathbb{R}^n - \{y\}$ be a homotopy of $\partial f, \partial g: \partial\Omega \to \mathbb{R}^n - \{y\}$ with $G(0,x) = \partial f(x)$ and $G(1,x) = \partial g(x)$. Define a map F on $(\{0,1\} \times \overline{\Omega}) \cup ([0,1] \times \partial\Omega)$ by $F(t,x) = G(t,x)$ for $(t,x) \in ([0,1] \times \partial\Omega)$ and for $x \in \overline{\Omega}$, $F(0,x) = f(x)$ and $F(1,x) = g(x)$. By the Stone-Weierstrass Theorem there is a sequence of polynomials $F_k: \overline{\Omega} \to \mathbb{R}^n$ converging uniformly to F on $(\{0,1\} \times \overline{\Omega}) \cup ([0,1] \times \partial\Omega)$. Define $Q_{k,t,t'}: [0,1] \times \overline{\Omega} \to \mathbb{R}^n$ for $t, t' \in [0,1]$ by $Q_{k,t,t'}(s,x) = sF_k(t,x) + (1-s)F_k(t',x)$. For each $t_0 \in [0,1]$ there is an $\epsilon > 0$ such that for $t, t' \in [t_0 - \epsilon, t_0 + \epsilon]$, we have $y \notin Q_{k,t,t'}([0,1] \times \partial\Omega)$ eventually, say for $k \geq L$. Let $t, t' \in [t_0 - \epsilon, t_0 + \epsilon]$. Then $Q_{k,t,t'}$ defines a smoothly in Ω mod $(\partial\Omega, y)$ homotopy of the polynomials $P_{k,t}, P_{k,t'}: \overline{\Omega} \to \mathbb{R}^n$, where $P_{k,t}(x) = F_k(t,x)$. By the compactness of $[0,1]$ the maps $P_{k,0}, P_{k,1}$ are mod $(\partial\Omega, y)$ homotopic. The conclusion follows from Theorem 3(2).

(3). Let $W_i \subset \Omega$ be domains containing A_i with $\overline{W}_i \cap f^{-1}(y) = A_i$ and with pairwise disjoint closures. Suppose $P_j: \overline{\Omega} \to \mathbb{R}^n$ is a sequence of polynomials converging uniformly to f such that for an integer L and $j \geq L$, $I(P_j, y) = I(f, y)$ and $I((P_j)_{\overline{W}_i}, y) = I(f_{\overline{W}_i}, y)$. By Theorem 3, $I(P_j, y) = \sum_i I((P_j)_{\overline{W}_i}, y)$ and thus $I(f, y) = \sum_i I(f_{\overline{W}_i}, y)$.

Corollary 5 extends Theorem A to continuous maps.

Corollary 5. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $f, g: \overline{\Omega} \to \mathbb{R}^n$ be continuous. Then $f^{-1}(O)$ and $g^{-1}(O)$ have the same number of components when counted with multiplicity if any one of the following is satisfied for $z \in \partial\Omega$.

i. $|f(z) + g(z)| < |f(z)| + |g(z)|$.
ii. $|f(z) - g(z)| < |f(z)|$.
iii. $Re(\overline{f(z)} \cdot g(z)) > 0$, where $\mathbb{R}^n \neq \mathbb{C}^k$.

Proof. If any one of the conditions is satisfied for $z \in \partial\Omega$, the vectors $g(z)$ and $f(z)$ are not collinear and so the map $F(t,z) = tf(z) + (1-t)g(z)$ defines a mod $(\partial\Omega, O)$ homotopy of f and g. Theorem 4 finishes the proof.

Corollary 6 extends Theorem A(1).

Corollary 6. Let Ω be a bounded domain in $\mathbb{C}^n$ containing O and let $f: \overline{\Omega} \to \mathbb{C}^n$ be a continuous map. If $Re(\overline{z} \cdot f(z)) > 0$ for $z \in \partial\Omega$, then $I(f, O) = 1$ and so $f^{-1}(O) \neq \emptyset$. In addition, if f is holomorphic in Ω, then $f^{-1}(O)$ is a singleton.

Proof. The first part is immediate from Corollary 5 and Theorem 4(1). The second part follows from the first and the fact that a compact analytic subset of $\mathbb{C}^n$ is a finite set and an order of multiplicity for a holomorphic map is nonnegative.

Professor Baker [1] presented examples of maps below where the number of solutions of $f(x) = y$ for a continuous map $f\colon B^2 \to \mathbb{R}^2$, $y \notin f(\partial B)$, is not exactly the absolute value, $|\gamma(f(\partial B^2), y)|$, of the winding number of the curve $f(\partial B^2)$ about y. Theorem 4 shows that if each connected component of the solution set is counted with multiplicity, the total number coincides with the winding number.

Examples. Let f, g, h be maps on the unit disk B^2 defined by $f(x,y) = (x, |y|)$, $g(x,y) = (x^2 - y^2, -2xy)$ and $h(x,y) = \phi(x^2+y^2)(x,y)$, where

$$\phi(t) = \begin{cases} 0, & \text{if } 0 \le t \le \frac{1}{2}; \\ 2t-1, & \text{if } \frac{1}{2} \le t \le 1 \ . \end{cases}$$

(1). The winding number $\gamma(f(\partial B^2), (0,0)) = 0$. The equation $f(x,y) = (0,0)$ has exactly one solution $(0,0)$ with $\mu_{(0,0)}(f) = 0$ and the equation $f(x,y) = (0, \frac{1}{2})$ has exactly two solutions with $\mu_{(0,\frac{1}{2})}(f) = 1$ and $\mu_{(0,-\frac{1}{2})}(f) = -1$.

(2). The winding number $\gamma(g(\partial B^2), (0,0)) = -2$. The equation $g(x,y) = (0,0)$ has exactly one solution $(0,0)$ with $\mu_{(0,0)}(g) = -2$ and the equation $g(x,y) = (\frac{1}{4}, 0)$ has exactly two solutions $(0,0)$ with $\mu_{(\frac{1}{2},0)}(g) = -1$ and $\mu_{(-\frac{1}{2},0)}(g) = -1$.

(3). The winding number $\gamma(h(\partial B^2), (0,0)) = 1$. The solution of the equation $h(x,y) = (0,0)$ is the connected set $A = \{(x,y) : |(x,y)| \le \frac{1}{2}\}$ with $\mu_A(h) = 1$.

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