REAL COMPACT OPERATORS IN FACTORS OF TYPE I, II, AND III, 0 < \lambda \leq 1

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Abstract. In the present paper the real ideals of relatively compact operators of $W^*$-algebras are considered. Similar to the complex case, a description (up to isomorphism) of the real two-sided ideal of relatively compact operators of the complex $W^*$-factors is given.

1. Introduction. Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space $H$. A weakly closed $*$-subalgebra $M$ with identity element $1$ in $B(H)$ is called a $W^*$-algebra. Let $P(M)$ be the set of all projections of $M$, $I$ be the ideal of all operators with the finite range projection relative to $M$, $J = T$ be the ideal of compact operators relatively to $M$. It is known [2], that $I$ and $J$ are proper if and only if $M$ is infinite; and that $J$ is the maximal two-sided ideal of $M$ without infinite projections. The compact operators relative to $M$ were defined by Sonis [6] (in the case of the algebras with Segal measure, i.e. for finite $W^*$-algebras) as the operators which are mapping bounded sets into relatively compact sets. In [4], an analogous notion of finiteness and compactness in purely infinite $W^*$-algebras was introduced and considered.

In the present paper we introduce and consider the ideal of compact operators relative to a real $W^*$-algebra. In a manner similar to the complex case, a description and classification (up to isomorphism) of the real two-sided ideal of the relatively compact operators is given.

2. Preliminary Information. A real $*$-subalgebra $R$ with $1$ in $B(H)$ is called a real $W^*$-algebra if it is closed in the weak operator topology and $R \cap iR = \{0\}$. A real $W^*$-algebra $R$ is called a real factor if its center $Z(R)$ contains only elements of the form $\{\lambda 1\}$, $\lambda \in \mathbb{R}$. We say that a real $W^*$-algebra $R$ is of the type $I_{fin}$, $I_{\infty}$, $I_1$, $II_\infty$, or $III_\lambda$ ($0 \leq \lambda \leq 1$) if the enveloping $W^*$-algebra $U(R) = R+iR$ has the corresponding type in the ordinary classification of $W^*$-algebras [1].

A linear mapping $\alpha$ with $\alpha(x^*) = \alpha(x)^*$ of the algebra $R$ into itself is called
- an $*$-automorphism if $\alpha(xy) = \alpha(x)\alpha(y)$;
- an $*$-antiautomorphism if $\alpha(xy) = \alpha(y)\alpha(x)$,
- an involutive if $\alpha^2(x) = \alpha(\alpha(x)) = x$, for all $x, y \in R$. 

1
A trace on a (complex or real) $W^*$-algebra $N$ is a linear function $\tau$ on the set $N^+$ of positive elements of $N$ with values in $[0, +\infty]$, satisfying the following condition:

$$\tau(uxu^*) = \tau(x),$$

for an arbitrary unitary $u$ and $x$ in $N$.

The trace $\tau$ is said to be finite if $\tau(1) < +\infty$; semifinite if given any $x \in N^+$ there is a nonzero $y \in N^+$, $y \leq x$ with $\tau(y) < +\infty$.

Let $R \subset B(H)$ be a real $W^*$-algebra, $M = R + iR$ be the enveloping $W^*$-algebra for $R$. Let $\tau$ be a semifinite trace on $R$. Subspace $K$ of $H$ with $K\eta R$, i.e. $P_K \in R$, is called

- finite relative to $\tau$ if $\tau(P_K) < \infty$, where $P_K$ projection of $H$ on $K$;
- compact relative to $\tau$ if $K$ is an approximate of the bounded sets from relatively finite subspaces.

Real operator $x$ of $H$ (i.e. $x \in R$) is called real compact relative to $\tau$ if it is the operator mapping bounded sets into relatively compact sets.

3. Compact Operators in Semifinite Real Factor. Let $I(R)$ be the set of all relatively compact operators of $R$.

**Theorem 1**. Let $R$ be a semifinite real factor. Then $I(R)$ is a unique (nonzero) uniformly closed two-sided ideal of $R$.


**Theorem 2**. Let $R$ be a semifinite real factor, $U = R + iR$ is its enveloping factor. Let $I(U)$ be a unique (nonzero) uniformly closed two-sided ideal of $U$. Then

$$I(U) = I(R) + iI(R).$$

**Proof**. Since $I(R)$ is a uniformly closed two-sided ideal, then $I(R) + iI(R)$ is also a uniformly closed two-sided ideal. In fact, let $\{c_n = a_n + i b_n\}$ be a Cauchy sequence in $I(R) + iI(R)$, i.e. $\|c_n - c_m\| \to 0$ as $n, m \to \infty$. Then $\|(a_n - a_m) + i (b_n - b_m)\| \to 0$ as $n, m \to \infty$. Using Lemma 1.1.3 (iii) from [1] we have

$$\max\{\|a_n - a_m\|, \|b_n - b_m\|\} \leq \|(a_n - a_m) + i (b_n - b_m)\|.$$

Therefore, $\|a_n - a_m\| \to 0$ and $\|b_n - b_m\| \to 0$ as $n, m \to \infty$. Thus, $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in $I(R)$. Hence, they converge to $a$ and $b$ in $I(R)$, respectively. Thus, $c_n = a_n + i b_n \to a + i b$ in $I(R) + iI(R)$, which is thus uniformly closed. Now, if $x = a + i b \in U$, $y = c + i d \in I(R) + iI(R)$, then $xy = (ac - bd) + i(ad + bc) \in I(R) + iI(R)$. Similarly, $yx \in I(R) + iI(R)$. Therefore, $I(R) + iI(R)$ is a uniformly closed two-sided ideal of $U$. Thus, we have proved that $I(R) + iI(R) \subset I(U)$. Now, since for $x \in I(U)$ we have $x = a + i b, a, b \in R$, let $I(U) = A + iB$, for some $A, B \subset R$. But, for $a \in A, b \in B$ we have $ab, ba \in I(U)$. Therefore, $ab, ba \in A$, hence, $A = B$, i.e.,
$I(U) = A+iA$. Then $I(R) \subset A$, $I(R) \subset R$. Let $\{a_n\}$ be a Cauchy sequence in $A \subset I(U)$. Since $I(U)$ is uniformly closed, $\{a_n\}$ converges to $a \in I(U)$. But, $R$ is also uniformly closed. Therefore, $a \in R$. Then $a \in A$. Now, let $x \in A, y \in R$. Since $I(U)$ is a two-sided ideal of $U$, $xy, yx \in I(U)$, i.e. $xy, yx \in A$ as $xy, yx \in R$. Therefore, $A$ is a uniformly closed two-sided ideal of $R$ with $I(R) \subset A$. Then by Theorem 1 we have $A = I(R)$. This completes the proof of the theorem.

4. Real Ideals of Compact Operators of Factors of Type III$_\lambda$, ($\lambda \neq 1$). Let us recall [3] the notion of the crossed product of a $W^*$-algebra by a locally compact topological group by its $^*$-automorphism. Let $N$ be a (complex or real) $W^*$-algebra in $B(H)$, $\gamma: G \rightarrow Aut(M)$ be a group homomorphism such that each map $g \rightarrow \gamma_g$ is strongly continuous. Let $L_2(G, H)$ be the Hilbert space of all $H$-valued square integrable functions on $G$. We consider a $^*$-algebra $U \subset B(L_2(G, H))$, generated by operators of the form: $\pi_\gamma(a)(a \in M)$ and $u(g)(g \in G)$, where

$$
(\pi_\gamma(a)\xi)(h) = \gamma_h^{-1}(a)\xi(h), \quad (u(g)\xi)(h) = \xi(g^{-1}h),
$$

$$
\xi = \xi(h) \in L_2(G, H), \ g, h \in G.
$$

The algebra $U$ is called crossed product of $M$ by $G$, and denoted by $W^*(M, G)$ (or $M \rtimes \gamma G$). Moreover, there exists a canonical embedding $\pi\gamma: M \rightarrow \pi_\gamma(M) \subset U$. Each element $x \in U$ has the form:

$x = \sum_{g \in G} \pi_\gamma(x(g))u(g)$, where $x(\cdot)$ is an $M$-valued function on $G$.

Let $\theta$ be a $^*$-automorphism of $N$. For the action $\{\theta^n\}$ of the group $Z$ on $N$ we denoted by $W^*(\theta, N)$, (or $N \rtimes \theta Z$) the crossed product of $N$ by $\theta$.

Now, let $R$ be a factor of type III$_\lambda$, ($\lambda \neq 1$). Then by [7], either

- there exist a real factor $F$ of type II$_\infty$ and an automorphism $\theta$ of $F$ such that $R$ is isomorphic to the crossed product $F \times_\theta Z$ or
- there exist a complex factor $N$ of a type II$_\infty$ and an antiautomorphism $\sigma$ of $N$ such that $R$ is isomorphic to $((N \oplus N^{op}) \times_\sigma Z, \beta)$, where $N^{op}$ is the opposite $W^*$-algebra for $N$, $\beta(x, y) = (y, x)$, for all $x, y \in N$.

In the first case, let $I(R)$ be the norm closure of $\text{span}\{x \in R^+ \mid E(x) \in I(F)\}$, where $E: R \rightarrow F$ is a unique faithful normal conditional expectation.

In the second case, let $I(R)$ be the norm closure of $\text{span}\{x \in R^+ \mid E(x) \in I(N \oplus N^{op})\}$, where $E$ is a unique faithful normal conditional expectation from $M$ to $N \oplus N^{op}$.

If we now apply Theorem 1 and use the scheme of proof of Theorem 6.2 from [4], then we prove a real analogue of the theorem of Halpern-Kaftal.

**Theorem 3.** In each case $I(R)$ is a unique (nonzero) uniformly closed two-sided ideal of $R$.

Similar to Theorem 2, we can prove the following theorem.
Theorem 4. Let $M$ be an injective factor of type $\text{III}_\lambda$, $0 < \lambda < 1$, $R$ and $Q$ are non-isomorphic real factors with the enveloping factor $M$, i.e. $R + iR = Q + iQ = M$. If $I(M)$ is a (nonzero) uniformly closed two-sided ideal of $M$, then

$$I(M) = I(R) + iI(R) \text{ and } I(M) = I(Q) + iI(Q),$$

where $I(R)$ and $I(Q)$ are non-isomorphic unique uniformly closed two-sided ideals of $R$ and $Q$, respectively.

5. Main Result. Let $M$ be a factor, $\alpha$ - an involutive $*$-antiautomorphism of $M$. Then from [1], the set $R = \{ x \in M : \alpha(x) = x^* \}$ is a real factor and the enveloping $W^*$-algebra $U(R)$ of $R$ coincides with $M$, and conversely, given an arbitrary real factor $R$ there exists a unique involutive $*$-antiautomorphism $\alpha_R$ of the $W^*$-algebra $U(R)$ such that $R = \{ x \in U(R) : \alpha(x) = x^* \}$. Moreover, $R_1$ and $R_2$ are two real $*$-isomorphic factors if and only if the enveloping factors $U(R_1)$ and $U(R_1)$ are $*$-isomorphic and the involutive $*$-antiautomorphism $\alpha_{R_1}$ and $\alpha_{R_2}$ are conjugate, i.e. $\alpha_{R_1} = \theta \cdot \alpha_{R_2} \cdot \theta^{-1}$, for some $*$-automorphism $\theta$.

It is known [1] that

- in factor of type $\text{I}_n$, $n$ even, there exists unique conjugacy class on involutive $*$-antiautomorphism;
- in factor of type $\text{I}_n$, $n$ odd or $n = \infty$, there exist exactly two conjugacy classes on involutive $*$-antiautomorphism;
- in injective factor of type $\text{II}_1$ there exists unique conjugacy class on involutive $*$-antiautomorphism;
- in injective factor of type $\text{II}_\infty$ there exists unique conjugacy class on involutive $*$-antiautomorphism;
- in injective factor of type $\text{III}_\lambda$, $0 < \lambda < 1$, there exist exactly two conjugacy classes on involutive $*$-antiautomorphism;
- in injective factor of type $\text{III}_1$ there exists unique conjugacy class on involutive $*$-antiautomorphism.

Hence, from Theorems 1 and 3 we obtain the following theorem.

Theorem 5. Let $M$ be a factor.

1) If $M$ has type $\text{I}_n$, $n$ even, then in $M$ there exist two (nonzero) uniformly closed two-sided real ideals up to isomorphisms;
2) If $M$ has type $\text{I}_n$, $n$ odd or $n = \infty$, then in $M$ there exist three (nonzero) uniformly closed two-sided real ideals up to isomorphisms;
3) If $M$ is an injective factor of type $\text{II}_1$ or type $\text{II}_\infty$, then in $M$ there exist two (nonzero) uniformly closed two-sided real ideals up to isomorphisms;
4) If $M$ is an injective factor of type $\text{III}_\lambda$ ($0 < \lambda < 1$), then in $M$ there exist three (nonzero) uniformly closed two-sided real ideals up to isomorphisms.
References


Mathematics Subject Classification (2000): 46L70

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