ON THE SOLUTION OF THE CUBIC PYTHAGOREAN DIOPHANTINE EQUATION $\mathbf{x^3 + y^3 + z^3 = a^3}$

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1. Introduction. For hundreds of years, mathematicians, including Euler, Vieta, Binet, Ramanujan and others have been fascinated and perplexed by the 3.1.3 cubic Pythagorean Diophantine equation, $x^3 + y^3 + z^3 = a^3$. Though complex partial solutions have been provided by these renowned mathematicians, a complete solution of this equation in integers still eludes us. In this paper, we propose to provide two independent relatively simple solutions of this intriguing equation. Though a complete general solution in integers is still not at hand, the solutions derived for the special cases described here do provide significant insight into this most fascinating equation and prove unequivocally that we can easily produce an infinite number of solutions in the family of equations satisfying the relationship $x^3 + y^3 + z^3 = a^3$.

2. First Solution. Consider the following subset of known solutions to the 3.1.3 cubic Pythagorean Diophantine equation, $x^3 + y^3 + z^3 = a^3$, in order to determine certain facts and relationships.

$$3^3 + 4^3 + 5^3 = 6^3$$

$$4^3 + 17^3 + 22^3 = 25^3$$

$$16^3 + 23^3 + 41^3 = 44^3$$

$$16^3 + 47^3 + 108^3 = 111^3$$

$$64^3 + 107^3 + 405^3 = 408^3$$

$$64^3 + 155^3 + 664^3 = 667^3.$$

Perhaps the most crucial insight of all is to initially rearrange the first equation in the above series to read $4^3 + 5^3 + 3^3 = 6^3$. This is a critical first step for two reasons. First, there are now two equations for each x value, and x is always a power of 4,

$$\text{i.e.,} \quad x = (4^n)^3 \quad \text{where} \quad n = 1, 2, 3, \ldots \qquad (1)$$

Secondly, in all cases shown,

$$a = z + 3. \tag{2}$$

Therefore, for each x value where $x_1 = x_2$, we have

$$x_1^3 + y_1^3 + z_1^3 = a_1^3$$

and

$$x_2^3 + y_2^3 + z_2^3 = a_2^3.$$

Since the x values shown are 4^1, 4^2, and 4^3, each cubed, the next x value (for the next pair of equations) must be $(4^4)^3 = 256^3$. Therefore,

$$256^3 + y_1^3 + z_1^3 = (z_1 + 3)^3$$

and

$$256^3 + y_2^3 + z_2^3 = (z_2 + 3)^3,$$

so

$$256^3 + y^3 = 9(z^2 + 3z + 3) \tag{3}$$

for each y and z.

Now, further study of the given equations reveals that for $n = 1$, $y_2 - y_1 = 12$; for $n = 2$, $y_2 - y_1 = 24$; and for $n = 3$, $y_2 - y_1 = 48$. Therefore, for $n = 4$, it is clear that $y_2 - y_1$ must equal 96.

Examination of the last equation above reveals that $(256^3 + y^3)$ must be evenly divisible by 9 since we are dealing here with whole numbers. Since 256^3 divided by 9 leaves a remainder of 1, y^3 divided by 9 must leave a remainder of 8.

Further study of the y values in the equations given reveals that for $n = 1$, the last digit of y_1 is 5 and of y_2 is 7; for $n = 2$, the last digit of y_1 is 3 and of y_2 is 7; for n=3, the last digits of y_1 and y_2 are 7 and 5, respectively. Therefore, it stands to reason that for $n = 4$ the last digits of y_1 and y_2 should be 7 and 3, respectively. At this point it must be understood that this fact has not yet been definitively proven, but it would appear to be a reasonable basis on which to proceed.

Now, since y is always greater than x, and y_1 and y_2 must end with the digits 7 and 3 in the pair of equations where $x = 256$, the first whole number to satisfy the conditions enumerated thus far would appear to be 257. However, in reviewing again equation (3),

$$256^3 + y^3 = 9(z^2 + 3z + 3),$$

if $y = 257$, then $\frac{(256^3+257^3)}{9}$ is, in fact, a whole number, but then z is not a whole number since, in solving the quadratic equation, the discriminant must be a perfect square to result in a whole number solution. Clearly, if we progressively increase y by multiples of 30, y will continue to meet all the required conditions (since 30^3, when divided by 9, will not change the remainder left by the previous number and therefore not change the last digit of the number for which we are searching which we have deduced to be a 7, whereas 10 or 20 added to the previous number and divided by 9 will change the remainder and therefore not result in a whole number since neither 10^3 nor 20^3, divided by 9, leaves a remainder of 0.) However, not until one lets $y = 467$ (which does equal 257 plus a multiple of 30) does z become a whole number.

As a result,

$$256^3 + 467^3 = 9(z^2 + 3z + 3),$$

so

$$z^2 + 3z - 13180528 = 0.$$

Therefore, $z = 3629$ and $a = 3632$, so

$$256^3 + 467^3 + 3629^3 = 3632^3.$$

(It also follows that z could equal -3632, in which case a would equal -3629, but since in all solutions derived in this manner the z and a values can be reversed simply by making them both negative, for discussion purposes in this paper, all such solutions will be ignored.)

Similarly:

$$256^3 + (467 + 96)^3 + z_2^3 = (z_2 + 3)^3$$

which can be solved to produce $z_2 = 4656$ and $a_2 = 4659$. Therefore,

$$256^3 + 563^3 + 4656^3 = 4659^3.$$

These solutions do verify when computed out. However, the shortcoming of this solution, though it is mathematically correct, is to determine the y values when $x = 4^5$, 4^6, etc., would still require a significant amount of trial and error and could not be nearly as easily or as readily solved since the y values (and therefore the z and a values) get much larger very quickly. This shortcoming can be overcome by the following mathematically more rigorous independent solution.

3. Second Solution. Recall that for each $n = 1, 2, 3, \ldots$ there is a set of two equations for each $x = (4^n)^3$ and in all cases $a = z + 3$. Recall also that for $x = 4$, $y_2 - y_1 = 12$; for $x = 16$, $y_2 - y_1 = 24$; for $x = 64$, $y_2 - y_1 = 48$; for $x = 256$, $y_2 - y_1 = 96, \ldots$. In fact, in all cases, it is clear that

$$y_2 - y_1 = 6\sqrt{x}. \tag{4}$$

However, it can also be seen from the first several solutions in this family of equations that for any given x value,

$$y_1 + y_2 = 4x + 6. \tag{5}$$

Therefore, the next set of solutions to satisfy $x^3 + y^3 + z^3 = a^3$ must be

$$256^3 + 467^3 + z_1^3 = (z_1 + 3)^3$$

and

$$256^3 + 563^3 + z_2^3 = (z_2 + 3)^3$$

since $467+563 = 4(256)+6$ and $563-467 = 96$ as required. These equations can then be solved in the usual way to obtain $z_1 = 3629$, $a_1 = 3632$ and $z_2 = 4656$, $a_2 = 4659$ which agree with my previous solution.

However, now we can easily progress to the next set of equations satisfying the given conditions by reference to $x = (4^5)^3 = 1024^3$. Since

$$y_1 + y_2 = 4(1024) + 6$$

and

$$y_2 - y_1 = 6\sqrt{1024} = 192,$$

$y_1 + y_2 = 4102$ and $y_2 - y_1 = 192$, so

$$1024^3 + 1955^3 + z_1^3 = (z_1 + 3)^3$$

and

$$1024^3 + 2147^3 + z_2^3 = (z_2 + 3)^3.$$

These equations can then be solved in the usual way to obtain $z_1 = 30813$ and $z_2 = 34912$. Therefore,

$$1024^3 + 1955^3 + 30813^3 = 30816^3$$

and

$$1024^3 + 2147^3 + 34912^3 = 34915^3.$$

This result is also readily confirmed by computation.

At this point, one could then continue to find an endless number of solutions to the 3.1.3 cubic Pythagorean Diophantine equation. In fact, with only the relationships derived thus far, no further insight is required to fully solve the equation as shown, as long as $x = 4^n$. However, curiously, further relationships between the variables can be deduced. So far, we have dealt only with the relationship between x and y and then used that information to solve for z and a in a standard fashion. But what of the relationship between x and z directly?

Rather intense scrutiny of the original solutions reveals that

$$z_1 + z_2 = 2\sqrt{x}(x+3) - 3. \tag{6}$$

Also,

$$z_2 - z_1 = 4x + 3. \tag{7}$$

With these relationships, one can easily and quickly determine z_1 and z_2 for any given x value. In fact, with the relationships now elucidated, all variables can be easily determined without even the use of a calculator.

For example: if $n = 6$,

$$x = (4^6)^3 = 4096^3,$$

$$y_2 - y_1 = 6\sqrt{4096} = 384,$$

from Equation (4), and

$$y_1 + y_2 = 4(4096) + 6 = 16390,$$

from Equation (5). Therefore,

$$2y_2 = 16774,$$

so

$$y_2 = 8387$$

and

$$y_1 = 8003.$$

Also,

$$z_1 + z_2 = 2\sqrt{4096}(4099) - 3 = 524669,$$

from Equation (6), and

$$z_2 - z_1 = 4(4096) + 3 = 16387$$

from Equation (7). Therefore,

$$2z_2 = 541056,$$

so

$$z_2 = 270528 \quad \text{and} \quad a_2 = 270531,$$

and

$$z_1 = 254141 \quad \text{and} \quad a_1 = 254144.$$

This yields

$$4096^3 + 8003^3 + 254141^3 = 254144^3$$

and

$$4096^3 + 8387^3 + 270528^3 = 270531^3.$$

With these relationships, we have now deduced direct relationships between x and (y_1 and y_2), between x and (z_1 and z_2) and between (y_1 and y_2) and (z_1 and z_2).

So, for example, if $n = 7$,

$$x = 4^7 = 16384,$$

$$y_1 = 32387,$$

$$y_2 = 33155,$$

$$z_1 = 2064765,$$

and

$$z_2 = 2130304.$$

Therefore,

$$16384^3 + 32387^3 + 2064765^3 = 2064768^3$$

and

$$16384^3 + 33155^3 + 2130304^3 = 2130307^3.$$

This then represents the simplest general solution to the 3.1.3 cubic Pythagorean Diophantine equation $x^3 + y^3 + z^3 = a^3$, where $x = 4^n$, in its most elegant form, and demonstrates the beauty and power of mathematical deduction and reasoning.

Not surprisingly, all the relationships between the variables as deduced in this paper for the subset of solutions of the 3.1.3 cubic Pythagorean Diophantine equation where $x = 4^n$ also hold true if n is equal to zero or a negative number. For example, it can be readily shown that for $n = 0$:

$$1^3 + 2^3 + (-1)^3 = 2^3$$

and

$$1^3 + 8^3 + 6^3 = 9^3,$$

and again the variables in these equations conform to all the relationships derived earlier in this paper. Furthermore, in all published discussions of the cubic Pythagorean Diophantine equation $x^3+y^3+z^3 = a^3$ that we have been able to find, including in Weisstein's Encyclopedia of Mathematics [1], two of the simplest examples shown are always

$$1^3 + 6^3 + 8^3 = 9^3$$

and

$$3^3 + 4^3 + 5^3 = 6^3$$

as opposed to

$$1^3 + 8^3 + 6^3 = 9^3$$

and

$$4^3 + 5^3 + 3^3 = 6^3;$$

this strongly implies that the authors have failed to grasp the significance of these equations being part of a subset of solutions where $x = 4^n$ and $a = z + 3$.

Similarly, where n equals a negative number, the following solutions, though obviously more unwieldy, continue to conform to all the relationships derived in this paper.

$$(4^{-1})^3 + 2^3 + \left(-\frac{15}{8}\right)^3 = \left(\frac{9}{8}\right)^3$$

$$(4^{-1})^3 + 5^3 + \left(\frac{17}{8}\right)^3 = \left(\frac{41}{8}\right)^3$$

$$(4^{-2})^3 + \left(\frac{19}{8}\right)^3 + \left(-\frac{151}{64}\right)^3 = \left(\frac{41}{64}\right)^3$$

$$(4^{-2})^3 + \left(\frac{31}{8}\right)^3 + \left(\frac{57}{64}\right)^3 = \left(\frac{249}{64}\right)^3$$

$$(4^{-3})^3 + \left(\frac{85}{32}\right)^3 + \left(-\frac{1359}{512}\right)^3 = \left(\frac{177}{512}\right)^3$$

$$(4^{-3})^3 + \left(\frac{109}{32}\right)^3 + \left(\frac{209}{512}\right)^3 = \left(\frac{1745}{512}\right)^3,$$

etc.

Finally, if one takes this solution one step further and specializes m to 2^n, based on the relationships proven in this paper, one can readily show that [3]

$$(m^2)^3 + (2m^2 - 3m + 3)^3 + (m^3 - 2m^2 + 3m - 3)^3 = (m^3 - 2m^2 + 3m)^3 \quad (8)$$

and

$$(m^2)^3 + (2m^2 + 3m + 3)^3 + (m^3 + 2m^2 + 3m)^3 = (m^3 + 2m^2 + 3m + 3)^3. \quad (9)$$

These equations represent the general solution of the 3.1.3 cubic Pythagorean Diophantine equation as derived in this paper and, in fact, hold true for any x value provided x is a perfect square, not only for those cases where $x = 4^n$. This perhaps ought to have been evident from the start since the initial x values shown, 4^3, 16^3, and 64^3, can each be easily expressed as $(2^n)^3$. So, for example, if $m = 1, 2, 4, 8$, etc., the equations that result will be encompassed by the previously outlined solution, but if, for example, $m = 3$ or 5 or 6, one obtains the following solutions, all of which continue to conform to all the relationships derived here:

$$9^3 + 12^3 + 15^3 = 18^3$$

(interesting arithmetic sequence)

$$9^3 + 30^3 + 54^3 = 57^3$$

$$25^3 + 38^3 + 87^3 = 90^3$$

$$25^3 + 68^3 + 190^3 = 193^3$$

$$36^3 + 57^3 + 159^3 = 162^3$$

$$36^3 + 93^3 + 306^3 = 309^3.$$

This brings us one step closer to the ultimate final general solution of the cubic Pythagorean Diophantine equation $x^3 + y^3 + z^3 = a^3$. However, there remain numerous other solutions to this intriguing equation where $x \neq 4^n$ and which are not encompassed by solutions derived by specializing m to 2^n, i.e. x is not a perfect square. For example:

$$7^3 + 14^3 + 17^3 = 20^3$$

$$11^3 + 15^3 + 27^3 = 29^3$$

$$26^3 + 55^3 + 78^3 = 87^3$$

$$28^3 + 53^3 + 75^3 = 84^3$$

$$33^3 + 70^3 + 92^3 = 105^3$$

$$149^3 + 256^3 + 363^3 = 408^3$$

(interesting arithmetic sequence of the x, y, and z terms.)

But none of these equations where x is not a perfect square fall within the realm of discussion of this paper. The general solutions and formulae derived herein apply only to the subset of solutions where x is a perfect square. However, what has been proven with the relationships derived in this paper is that where x is a perfect square, there always exist two and only two solutions for the cubic Pythagorean Diophantine equation $x^3 + y^3 + z^3 = a^3$ for each x value and one can easily deduce both solutions for any given x value. So, for example, if we randomly choose an x value of 32041 (179^2), one readily obtains

$$32041^3 + 63548^3 + 5671791^3 = 5671794^3$$

and

$$32041^3 + 64622^3 + 5799958^3 = 5799961^3.$$

It is also of passing interest to note that where n is a positive number beginning with $n = 1$, for those solutions where $x = 4^n$, there does exist a periodicity to the last digit of the y values (5-7, 3-7, 7-5, 7-3, then repeat) and z values (3-2, 1-8, 5-4, 9-6, then repeat) and therefore the a values (since $a = z + 3$). This confirms the assumption made in the first solution outlined in this paper.

4. Conclusion. Two independent general solutions are provided for the cubic Pythagorean Diophantine equation $x^3 + y^3 + z^3 = a^3$ for those cases where x is a perfect square. Though the relationships and solutions derived apply only to the subset of cases as described, they do allow one to easily derive a theoretically unlimited number of solutions to this equation. At this point in time, Dickson's statement made years ago still holds true: "No general solution giving all positive integral solutions [to the equation $x^3 + y^3 + z^3 = a^3$] is known."[2] Perhaps future generations of mathematicians, besides, of course, finding the ultimate general solution in integers which encompasses all known solutions, ought to endeavour to determine why all the interrelationships which have been deduced exist between the variables in this subset of solutions to this fascinating equation. In fact, further more complex relationships between the variables can be elucidated (see Appendix 1), but all the relationships deduced apply only to the subset of solutions where x is a perfect square. Furthermore, all the relationships are of a horizontal nature (between the variables in a given equation or set of equations with the same x value). Could relationships of a vertical nature exist (between the variables in one set of equations with a given x value and the variables in the set of equations with an "n" value one more or one less)? This may or may not prove to be a fruitful approach for further research.

Finally, though none of the solutions provided here or elsewhere results in the all-encompassing general solution of all known solutions in integers to the cubic Pythagorean Diophantine equation $x^3 + y^3 + z^3 = a^3$, we believe that the uniqueness of the solutions derived here is related to their relative simplicity, based on insights and relationships as described, as compared to solutions derived by renowned mathematicians in the past.

Acknowledgements. Appreciation is extended to Professor Cameron Stewart at the Department of Pure Mathematics of the University of Waterloo, Waterloo, Ontario, Canada, and to Ms. Roberta Massiah of Washington, D.C., U.S.A. for their guidance and encouragement during the course of this work, as well as to Ms. Savannah Gomes, a Ph.D. candidate in the Department of Mathematics at the University of Toronto, Toronto, Ontario, Canada for her assistance in the preparation of the manuscript.

Addendum. Further review of the family of solutions of the cubic pythagorean diophantine equation $x^3 + y^3 + z^3 = a^3$ where x is a perfect square, as derived in this paper, or alternatively, a review of equations (8) and (9), which represent the general solutions of this subset of solutions, reveals that

$$y_1 + z_1 = x^{\frac{3}{2}} \tag{1}$$

and

$$a_2 - y_2 = x^{\frac{3}{2}}. \tag{2}$$

These relationships may be of particular interest in the ultimate general solution of this cubic diophantine equation since they are the only relationships derived by the author in this paper which do not simultaneously involve variables from each of the two equations in each pair of equations with the same x value; furthermore, each of the relationships equate to the same value of $x^{\frac{3}{2}}$ which, when squared, produces x^3, the first term in the diophantine equation that we are trying to solve. It is also readily evident that

$$y_1 + y_2 + z_1 = a_2 \tag{3}$$

for any given x value provided x is a perfect square. Whether these relationships will generate the necessary impetus to allow us to incorporate the solutions provided here into an all-encompassing solution which specifically encompasses those cases where x is not a perfect square remains to be seen but will require continued effort and research.

References

1. *CRC Concise Encyclopedia of Mathematics,* 2nd ed., Eric W. Weisstein, Editor, CRC Press, Boca Raton, FL, 2002, 642–646.
2. L. E. Dickson, *History of the Theory of Numbers,* Vol. 2: Diophantine Analysis, Chelsea, New York, 1966.
3. C. Stewart, Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada, Personal Communication.

Appendices.

A. Further Mathematical Relationships

$x^3 + y^3 + z^3 = a^3$; x is a perfect square.

Given (4)

$$y_2 - y_1 = 6\sqrt{x}$$

and (5)

$$y_2 + y_1 = 4x + 6,$$

yields

$$y_2 = 2x + 3 + 3\sqrt{x} \tag{10}$$

and

$$y_1 = 2x + 3 - 3\sqrt{x}. \tag{11}$$

Also, from (5) and (7)

$$z_2 - z_1 = 4x + 3 = y_1 + y_2 - 3. \tag{12}$$

Now, using (6)

$$z_1 + z_2 = 2\sqrt{x}(x+3) - 3$$

so

$$z_2 = 2x + \sqrt{x}(x+3) \tag{13}$$

and

$$z_1 = -2x - 3 + \sqrt{x}(x+3). \tag{14}$$

Now, (6)

$$z_1 + z_2 = 2\sqrt{x}(x+3) - 3$$

and (12)

$$z_2 - z_1 = y_1 + y_2 - 3,$$

give

$$z_2 = \sqrt{x}(x+3) - 3 + \left(\frac{y_1 + y_2}{2}\right), \tag{15}$$

but (13) is

$$z_2 = 2x + \sqrt{x}(x+3)$$

and (5) gives

$$x = \frac{y_1 + y_2 - 6}{4}. \tag{16}$$

Therefore, from (13) and (14),

$$z_2 = \frac{y_1 + y_2 - 6}{2} + \left(\frac{\sqrt{y_1 + y_2 - 6}}{2}\right)\left(\frac{y_1 + y_2 + 6}{4}\right) \tag{17}$$

and

$$z_1 = \frac{y_1 + y_2 - 6}{2} + \left(\frac{\sqrt{y_1 + y_2 - 6}}{2}\right)\left(\frac{y_1 + y_2 + 6}{4}\right) - y_1 - y_2 + 3. \tag{18}$$

Also, since (11),

$$y_1 = 2x + 3 - 3\sqrt{x},$$

and (10),

$$y_2 = 2x + 3 + 3\sqrt{x},$$

and (16),

$$x = \frac{y_1 + y_2 - 6}{4},$$

and (14),

$$z_1 = \sqrt{x}(x+3) - 2x - 3,$$

and (13),

$$z_2 = \sqrt{x}(x+3) + 2x,$$

we see that

$$z_1 = \left(\frac{\sqrt{y_1 + y_2 - 6}}{2}\right)\left(\frac{y_1 + y_2 + 6}{4}\right) - \left(\frac{y_1 + y_2}{2}\right) \tag{19}$$

and

$$z_2 = \left(\frac{\sqrt{y_1 + y_2 - 6}}{2}\right)\left(\frac{y_1 + y_2 + 6}{4}\right) + \left(\frac{y_1 + y_2}{2}\right) - 3. \tag{20}$$

Therefore, as previously stated

$$z_1 + z_2 = 2\sqrt{x}(x+3) - 3$$

and

$$z_2 - z_1 = 4x + 3 = y_1 + y_2 - 3.$$

B. Examples of Solutions.

$x^3 + y^3 + z^3 = a^3$; x is a perfect square.

$$1^3 + 2^3 + (-1)^3 = 2^3$$

$$1^3 + 8^3 + 6^3 = 9^3$$

$$4^3 + 5^3 + 3^3 = 6^3$$

$$4^3 + 17^3 + 22^3 = 25^3$$

$$9^3 + 12^3 + 15^3 = 18^3$$

$$9^3 + 30^3 + 54^3 = 57^3$$

$$16^3 + 23^3 + 41^3 = 44^3$$

$$16^3 + 47^3 + 108^3 = 111^3$$

$$25^3 + 38^3 + 87^3 = 90^3$$

$$25^3 + 68^3 + 190^3 = 193^3$$

$$36^3 + 57^3 + 159^3 = 162^3$$

$$36^3 + 93^3 + 306^3 = 309^3$$

$$49^3 + 80^3 + 263^3 = 266^3$$

$$49^3 + 122^3 + 462^3 = 465^3$$

$$64^3 + 107^3 + 405^3 = 408^3$$

$$64^3 + 155^3 + 664^3 = 667^3$$

$$81^3 + 138^3 + 591^3 = 594^3$$

$$81^3 + 192^3 + 918^3 = 921^3$$

$$100^3 + 173^3 + 827^3 = 830^3$$

$$100^3 + 233^3 + 1230^3 = 1233^3,$$

etc.

Mathematics Subject Classification (2000): 11D25

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