SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

152. [2004, 130; 2005, 201–202] *Proposed by Joe Flowers and Doug Martin (student), Texas Lutheran University, Seguin, Texas.*

Let

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

denote the Laplace transform of $f(t)$. Find $L[\sin^n bt]$, where b is any real constant and n is any non-negative integer.

Also solved by Kenneth B. Davenport, Dallas, Pennsylvania.

153. [2005, 52] *Proposed by Joe Howard, Portales, New Mexico.*

Let $n \geq 2$ be an integer. Prove that

$$n^n > (n+1)^{n-1} + \frac{n}{n+1}.$$

Solution by Thomas P. Dence, Ashland University, Ashland, OH. The inequality is clearly true for $n = 2$. Since

$$\frac{n}{n+1} < 1$$

and $n^n, (n+1)^{n-1} \in \mathbb{N}$, then it suffices to show that

$$n^n \geq (n+1)^{n-1} + 1$$

or equivalently,

$$n^n > (n+1)^{n-1}$$

for all $n \geq 3$. But this is true if and only if

$$\frac{n^n}{(n+1)^{n-1}} > 1$$

if and only if

$$n+1 > \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n.$$

Expanding the latter binomial gives

$$\left(1+\frac{1}{n}\right)^n = 1+1+\binom{n}{2}\frac{1}{n^2}+\binom{n}{3}\frac{1}{n^3}+\cdots+\binom{n}{n}\frac{1}{n^n}$$

$$< 1+1+\frac{1}{2!}\frac{n-1}{n}+\frac{1}{3!}\frac{n-1}{n}\frac{n-2}{n}+\cdots+\frac{1}{n!}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{1}{n}$$

$$< \sum_{k=0}^{n}\frac{1}{k!} < 1+\sum_{k=1}^{\infty}\frac{1}{k!} < 1+\sum_{k=0}^{\infty}\frac{1}{2^k} = 3 < n+1.$$

for all $n \geq 3$.

Also solved by Kenneth B. Davenport, Dallas, Pennsylvania; Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan; Mohamed Akkouchi, Université Cadi Ayyad, Marrakech, Morocco (2 solutions); C. Wesley Nevans, Truman State University, Kirksville, Missouri (student); Huizeng Qin, Shandong University of Technology, Shandong, People's Republic of China; Said Amghibech, Sainte Foy (qc), Canada; and the proposer.

154. [2005, 52] *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan.*

Let $n \geq 1$ be an integer and $m > 0$, $m < 2n$. Prove that

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\sin x_1 \sin x_2 \cdots \sin x_n}{(x_1 + x_2 + \cdots + x_n)^m} dx_1 dx_2 \cdots dx_n = \frac{1}{2\Gamma(m)} \cdot B\left(\frac{m}{2}; n - \frac{m}{2}\right),$$

where $\Gamma(\cdot)$ is the Gamma function and $B(\cdot, \cdot)$ is the Beta function.

Solution by Said Amghibech, Sainte Foy (qc), Canada . We have

$$\Gamma(m) = \int_0^\infty t^{m-1} \exp(-t) dt = a^m \int_0^\infty t^{m-1} \exp(-at) dt$$

for all $a > 0$. By choosing $a = x_1 + x_2 + \cdots + x_n$ we get

$$\Gamma(m) \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\sin x_1 \sin x_2 \cdots \sin x_n}{(x_1 + x_2 + \cdots + x_n)^m} dx_1 dx_2 \cdots dx_n =: I$$

and

$$I = \int_0^\infty \left(\int_0^\infty \sin x \exp(-tx) dx \right)^n t^{m-1} dt = \int_0^\infty \frac{t^{m-1}}{(1+t^2)^n} dt.$$

By putting $\tan s = t$ we obtain

$$I = \int_0^{\pi/2} \cos^{2n-m-1} s \sin^{m-1} s ds.$$

By putting $\sin^2 s = y$ we get

$$2I = \int_0^1 y^{\frac{m}{2}-1} (1-y)^{n-\frac{m}{2}-1} dy = B\left(\frac{m}{2}, n - \frac{m}{2}\right)$$

which gives the result.

Also solved by the proposer.

155. [2005, 53] *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Let P be any point inside $\triangle ABC$ and let $A' = AP \cap BC$, $B' = BP \cap AC$, and $C' = CP \cap AB$. Prove that

$$\frac{1}{AA'^2} + \frac{1}{BB'^2} + \frac{1}{CC'^2} \geq \frac{4\sqrt{3}}{3}\sqrt{\frac{1}{AP^4 + BP^4 + CP^4}}.$$

Solution by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan. First we show that

$$\frac{AP}{AA'} + \frac{BP}{BB'} + \frac{CP}{CC'} = 2.$$

To see this we notice that

$$\frac{AP}{AA'} = 1 - \frac{PA'}{AA'} = 1 - \frac{\sigma[BPC]}{\sigma[ABC]},$$

where $\sigma[T]$ is the area of triangle T. Let $E \in BC$ such that $PE \perp BC$ and $D \in BC$ such that $AD \perp BC$. Then $\triangle PEA' \sim \triangle ADA'$ implies that

$$\frac{PA'}{AA'} = \frac{PE}{AD} = \frac{PE \cdot BC \cdot \frac{1}{2}}{AD \cdot BC \cdot \frac{1}{2}} = \frac{\sigma[BPC]}{\sigma[ABC]}.$$

Therefore,

$$\frac{AP}{AA'} + \frac{BP}{BB'} + \frac{CP}{CC'} = 3 - \frac{PA'}{AA'} - \frac{PB'}{BB'} - \frac{PC'}{CC'}$$

$$= 3 - \frac{\sigma[BPC] + \sigma[PCA] + \sigma[APB]}{\sigma[ABC]} = 3 - 1 = 2.$$

Next we notice that

$$AP^4 + BP^4 + CP^4 \geq \frac{(AP^2 + BP^2 + CP^2)^2}{3}$$

since

$$a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3}.$$

Therefore,

$$\frac{4\sqrt{3}}{3}\cdot\sqrt{\frac{1}{AP^4+BP^4+CP^4}} \leq \frac{4\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{AP^2+BP^2+CP^2}$$

$$=\frac{4}{AP^2+BP^2+CP^2}.$$

Let us denote

$$AA'=x;\quad BB'=y;\quad CC'=z;\quad \frac{AP}{AA'}=\alpha;\quad \frac{BP}{BB'}=\beta;\quad \frac{CP}{CC'}=\gamma.$$

We observe then that it suffices to show

$$\frac{4}{AP^2+BP^2+CP^2} \leq \frac{1}{AA'^2}+\frac{1}{BB'^2}+\frac{1}{CC'^2}.$$

In view of the above notations this inequality is equivalent to

$$\frac{4}{\alpha^2x^2+\beta^2y^2+\gamma^2z^2} \leq \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$$

We know that

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{\alpha^2}{\alpha^2x^2}+\frac{\beta^2}{\beta^2y^2}+\frac{\gamma^2}{\gamma^2z^2}.$$

In view of the Cauchy-Buniakowsky-Schwartz inequality we get that

$$(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)\cdot\left(\frac{\alpha^2}{\alpha^2x^2}+\frac{\beta^2}{\beta^2y^2}+\frac{\gamma^2}{\gamma^2z^2}\right)$$

$$\geq\left(\alpha x\cdot\frac{1}{x}+\beta y\cdot\frac{1}{y}+\gamma z\cdot\frac{1}{z}\right)^2$$

$$=(\alpha+\beta+\gamma)^2=4.$$

Therefore,

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geq\frac{4}{\alpha^2x^2+\beta^2y^2+\gamma^2z^2}.$$

Also solved by Huizeng Qin, Shandong University of Technology, Shandong, People's Republic of China and the proposer.

156. [2005, 53] *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan.*

Prove that

$$I=\int_0^{\frac{\pi}{2}}\cos^2 x\left(\ln^2(2\cos x)-\ln(2\cos x)\right)dx=\frac{\pi^3}{48}-\frac{\pi}{4}.$$

Solution by Joe Howard, Portales, New Mexico. From [1],

$$\int_0^{\frac{\pi}{2}}\ln^2(2\sin\theta)d\theta=\frac{\pi^3}{24}.$$

By letting $\theta=\frac{\pi}{2}-x$,

$$\int_0^{\frac{\pi}{2}}\ln^2(2\cos x)dx=\frac{\pi^3}{24}. \tag{1}$$

By L'Hopital's Rule

$$\lim_{x\to\frac{\pi}{2}}\sin 2x\ln^k(2\cos x)=0$$

for $k=1,2$. Hence for $k=1,2$,

$$\sin 2x\ln^k(2\cos x)\Big|_0^{\frac{\pi}{2}}=0. \tag{2}$$

Integrating by parts where $u = x$

$$\int_0^{\frac{\pi}{2}} x \cdot \tan x \cdot \ln(2\cos x)dx = -\frac{x}{2}\ln^2(2\cos x)\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2(2\cos x)dx. \quad (3)$$

Integrating by parts ($u = \ln^2(2\cos x)$) and using (1), (2), and (3),

$$\begin{aligned} I_1 &= \int_0^{\frac{\pi}{2}} \cos^2 x \ln^2(2\cos x)dx \\ &= \frac{1}{4}\sin 2x \ln^2(2\cos x)\Big|_0^{\frac{\pi}{2}} \\ &+ \left(\frac{1}{2}x\ln^2(2\cos x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} x\tan x \ln(2\cos x)dx\right) \\ &+ \int_0^{\frac{\pi}{2}} \sin^2 x \ln(2\cos x)dx \\ &= 0 + \frac{\pi^3}{48} + \int_0^{\frac{\pi}{2}} \sin^2 x \ln(2\cos x)dx. \end{aligned}$$

Integrating by parts ($u = \ln(2\cos x)$) and using (2),

$$\begin{aligned} I_2 &= \int_0^{\frac{\pi}{2}} \cos 2x \ln(2\cos x)dx = \frac{1}{2}\sin 2x \ln(2\cos x)\Big|_0^{\frac{\pi}{2}} \\ &+ \int_0^{\frac{\pi}{2}} \sin^2 x dx = 0 + \frac{\pi}{4}. \end{aligned}$$

Finally,

$$\begin{aligned} I &= I_1 - \int_0^{\frac{\pi}{2}} \cos^2 x \ln(2\cos x)dx \\ &= \frac{\pi^3}{48} + \int_0^{\frac{\pi}{2}} (\sin^2 x - \cos^2 x)\ln(2\cos x)dx \\ &= \frac{\pi^3}{48} - I_2 = \frac{\pi^3}{48} - \frac{\pi}{4}. \end{aligned}$$

Reference

1. F. Bowman, "Note on the Integral $\int_0^{\frac{\pi}{2}} (\ln \sin \theta)^n \, d\theta$," *Journal of the London Mathematical Society,* 22 (1947), 172.

Also solved by Kenneth B. Davenport, Dallas, Pennsylvania; Huizeng Qin and Yousim Lu (jointly), Shandong University of Technology, Shandong, People's Republic of China; and the proposer.