THE DETERMINANT OF THE WHEEL GRAPH AND CONJECTURES BY YONG

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Abstract. We prove that the determinant of the adjacency matrix of the wheel graph of even order is equal to the determinant of the adjacency matrix of the complete graph of the same order. We present counterexamples to two conjectures by Yong about simple undirected graphs.

1. Introduction. In this paper, the word graph refers to a simple undirected graph. We use K_n to refer to the complete graph on n vertices, C_n to refer to the cycle on n vertices ($n \geq 3$), and W_n to refer to the wheel on n vertices ($n \geq 4$). We recall that $W_n = C_{n-1} + K_1$. We denote the complement of a graph G by $\overline{G}$ and the adjacency matrix of G by $A(G)$. By writing G_n, we are denoting a graph having n vertices.

Yong [7] proposed three conjectures about graphs. The first is:

<u>C1</u>. The graph G_n is complete if and only if $\det(A(G_n)) = (-1)^{n-1}(n-1)$.

While Olesky, Driessche, and Verner [4] have disproved this conjecture, their paper focused on finding a non-complete graph whose determinant is equal to $\det(A(K_n))$. This paper demonstrates that $\det(A(K_n)) = \det(A(W_n))$ for n even and 4 or larger, thereby providing (for n even and greater than 4) a family of counterexamples to C1.

Jinsong Tam [1] noticed that Yong's second conjecture follows from a theorem by Collatz-Singowitz. Yong's third conjecture is:

<u>C3</u>. Let $n \geq 4$, and let $\lambda_m(G_n)$ be the m-th largest eigenvalue of G_n. Then $\lambda_k(G_n) = -1$ for $2 \leq k \leq [n/2]$ implies that $\lambda_j(G_n) = -1$, for $k \leq j \leq n-k+2$.

Stevanovic [6] disproved C3 using graphs having 9 vertices. Here, using C_n and K_n we provide new counterexamples to C3 by constructing a family of graphs for which the number of vertices is not restricted.

As an aid to forthcoming calculations, we assume the vertices of C_n are labeled as $v_1, v_2, \ldots, v_n$ with v_k adjacent to v_{k+1} for $1 \leq k \leq n-1$ and with v_n adjacent to v_1. We assume the vertices of W_n are labeled as $v_1, v_2, \ldots, v_n$ with v_k adjacent to v_{k+1} for $2 \leq k \leq n-1$, with v_n adjacent to v_2, and with v_1 adjacent to all other vertices.

2. The Wheel Graph and Yong's First Conjecture. The cycle graph C_n has the following properties [2, 3]:

Since $A(C_n)$ is circulant, then its eigenvalues are given by $\lambda_k = e^{2\pi ki/n} + e^{2(n-1)\pi ki/n}$, $k = 1, \ldots, n$, where $i = \sqrt{-1}$. In particular, 2 is the largest eigenvalue of $A(C_n)$. The $n \times 1$ constant vector of 1's is a corresponding eigenvector. For every eigenvector corresponding to any other eigenvalue, the sum of the entries of the eigenvector is zero. If n is odd, then every eigenvalue is repeated except the eigenvalue 2, and the product of the eigenvalues except the eigenvalue 2 is equal to 1. When n is even, -2 is the smallest eigenvalue of $A(C_n)$. In this case, every eigenvalue of $A(C_n)$ is repeated except the eigenvalues 2 and -2, and, for $n \bmod 4$ not zero, the product of the eigenvalues, except for 2 and -2, is 1. If $n \bmod 4$ is zero, then $A(C_n)$ is singular.

Theorem 2.1. Let $n \geq 4$,

$$\alpha = \frac{1+\sqrt{n}}{n-1}, \quad \beta = \frac{1-\sqrt{n}}{n-1}, \quad a = \begin{bmatrix} 1 \\ \alpha \\ \alpha \\ \vdots \\ \alpha \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 1 \\ \beta \\ \beta \\ \vdots \\ \beta \end{bmatrix}.$$

Then, the eigenvalues of $A(W_n)$ are $1 \pm \sqrt{n}$ together with the eigenvalues of $A(C_{n-1})$ except the eigenvalue 2. Moreover, a is an eigenvector of $A(W_n)$ corresponding to the eigenvalue $1+\sqrt{n}$ and b is an eigenvector of $A(W_n)$ corresponding to the eigenvalue $1-\sqrt{n}$. Also, if v is an eigenvector of $A(C_{n-1})$ corresponding to the eigenvalue λ, where λ is any one of the $n-2$ eigenvalues shared by $A(C_{n-1})$ and $A(W_n)$, then

$$\begin{bmatrix} 0 \\ v \end{bmatrix}$$

is an eigenvector of $A(W_n)$ corresponding to the eigenvalue λ.

Proof. Let $m = n-1$. Then, it is easy to verify that $(1+\sqrt{n}, a)$ and $(1-\sqrt{n}, b)$ are eigenpairs of $A(W_n)$. Now note that for any eigenvector of $A(C_{n-1})$ corresponding to an eigenvalue different than 2, the sum of the entries of the eigenvector is 0. Let (λ, v) be an eigenpair of $A(C_{n-1})$ where $\lambda \neq 2$ and let

$$z = \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

When we multiply the first row of $A(W_n)$ with z we get $\sum_{k=1}^{n-1} v_k$ which is zero. The product of row k, $k \neq 1$, of $A(W_n)$ with z is equal to the product of row $k-1$ of $A(C_{n-1})$ with v. Thus, $A(W_n)z = \lambda z$. Since the eigenvectors of $A(C_{n-1})$ are known, then those $n-2$ eigenvectors of $A(W_n)$ are known.

The facts in the above theorem can also be found in [5]. However, unlike [5], in the proof of the theorem we depended on the fact that the eigenstructure of the adjacency matrix of the cycle graph is known because it is circulant. We note that determinants were not discussed in [5].

Note that the eigenvalues of $A(W_n)$ are $1+\sqrt{n}$, $1-\sqrt{n}$, $e^{2\pi ki/m} + e^{2(m-1)\pi ki/m}$, $k = 1, \ldots, m-1$, where $m = n-1$. Thus, $A(C_{n-1})$ and $A(W_n)$ share $n-2$ eigenvalues, and $n-2$ of the eigenvectors of $A(W_n)$ can be obtained from $n-2$ eigenvectors of $A(C_{n-1})$. In the following corollary, we use the eigenvalues of the adjacency matrix of the cycle graph to find the determinant of the adjacency matrix of the wheel graph.

Corollary 2.2. Let $n \geq 4$. Then

$$\det(A(W_n)) = \begin{cases} 2(n-1), & \text{if } n \bmod 4 \text{ is } 3 \\ 0, & \text{if } n \bmod 4 \text{ is } 1 \\ 1-n, & \text{if } n \bmod 4 \text{ is 2 or 0.} \end{cases}$$

Proof. Let $m = n-1$. If n is even, then m is odd, and hence, the product of the eigenvalues of $A(C_m)$ except the eigenvalue 2 is 1. Therefore, the product of the eigenvalues of $A(W_n)$ except the eigenvalues $1+\sqrt{n}$ and $1-\sqrt{n}$ is 1. Hence, $\det(A(W_n)) = 1-n$. Now if n is odd, then m is even, and hence, the product of the eigenvalues of $A(C_m)$ except the eigenvalue 2 is -2 if $m \bmod 4$ is 2 and the product is 0 if $m \bmod 4$ is 0. Therefore, the product of the eigenvalues of $A(W_n)$ except the eigenvalues $1+\sqrt{n}$ and $1-\sqrt{n}$ is -2 if $n \bmod 4$ is 3 and the product is 0 if $n \bmod 4$ is 1.

Thus, if n is an even integer greater than 2, then $\det(A(W_n)) = \det(A(K_n))$. Therefore, any member of the family $\{W_n \mid n \in 2\mathbb{Z},\ n > 4\}$ provides a counterexample to Yong's first conjecture.

3. Yong's Third Conjecture. We present new counterexamples to C3. Unlike the graphs mentioned in [6], the graphs we present are not restricted to a limited number of vertices. Let

$$M = \begin{bmatrix} A(K_{15}) & A(\overline{C_{15}}) \\ A(\overline{C_{15}}) & A(C_{15}) \end{bmatrix}.$$

Then the eigenvalues of M are: 21.41641, 3.57433, 3.57433, 2.78339, 2.78339, 1.61803, 1.61803, 0.27977, 0.27977, -0.40898, -0.40898, -0.61803,

-0.61803, -1.00000, -1.00000, -1.00000, -1.00000, -1.48883, -1.48883, -2.00000, -2.00000, -2.00000, -2.00000, -2.44512, -2.44512, -2.54732, -2.54732, -2.74724, -2.74724, -5.41641. Now let G be the graph whose adjacency matrix is M. Taking $k = 14$ in the conjecture, implies $n-k+2 = 18$. According to the conjecture, $\lambda_j(G)$, $j = 14, \ldots, 18$, must be -1. Clearly, this is not the case, since $\lambda_{18}(G) \neq -1$. Note that it is easy to verify (e.g. by Gaussian Elimination) that the rank of $M+I$, where I is the identity matrix of order 30, is 26. Note also that G is connected since the first vertex is adjacent to all other vertices except vertices 16, 17, and 30. But, vertex 1 is adjacent to vertex 4 which is adjacent to vertices 16, 17, and 30.

Apparently, if $m \in 3\mathbb{Z}^+$ and $m \geq 9$, then every graph G whose adjacency matrix is

$$M = \begin{bmatrix} A(K_m) & A(\overline{C_m}) \\ A(\overline{C_m}) & A(C_m) \end{bmatrix},$$

provides a counterexample to C3. We observed that for such values of m, $\lambda_j(G) = -1$ for $j = m-1, m, m+1, m+2$, and the rank of $I+M$ is $2m-4$, where I is the identity matrix of order $2m$.

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