DIOPHANTINE EQUATIONS, FIBONACCI HYPERBOLAS, AND QUADRATIC FORMS

Keith Brandt and John Koelzer

1. Introduction. In *Mathematical Diversions* [4], Hunter and Madachy ask for the ages of a boy and his mother, given the following:

His mother's age and his, when multiplied together, come to one more than the square of their difference.

We used this problem in a contest for our students. While reviewing the solutions submitted, we discovered some oddities about this question and its corresponding Diophantine equation. All integer solutions to the equation, and those of an incorrect formulation of the equation, consist of Fibonacci numbers. Furthermore, there is an interesting duality between the solutions to the correct and incorrect formulations.

2. The Equations. Let x and y represent the ages of the boy and his mother, respectively. The question leads to the equation

$$xy = 1 + (x - y)^2,$$

which simplifies to

$$x^2 - 3xy + y^2 = -1. \tag{1}$$

One of the solutions submitted had the 1 on the wrong side of the equation,

$$xy + 1 = (x - y)^2,$$

which simplifies to

$$x^2 - 3xy + y^2 = 1. \tag{2}$$

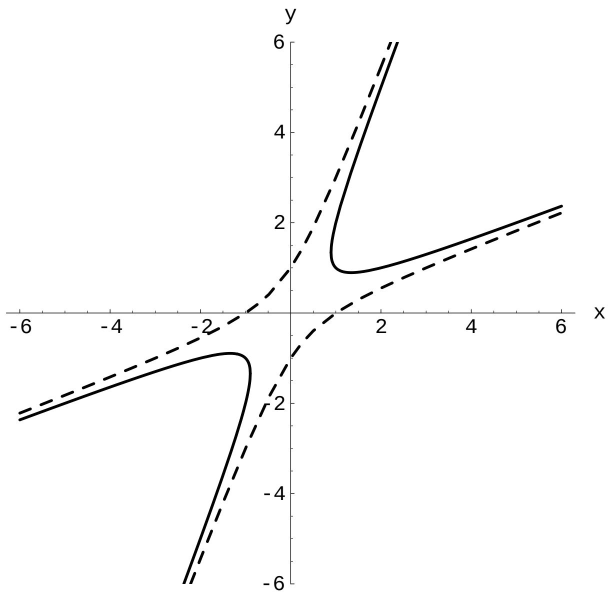

Figure 1. The curves defined by $xy = 1 + (x - y)^2$ (solid) and $xy + 1 = (x - y)^2$ (dashed).

The substitutions $u = 2x - 3y$ and $v = y$ reduce equations (1) and (2), respectively, to

$$u^2 - 5v^2 = -4 \tag{3}$$

and

$$u^2 - 5v^2 = 4, \tag{4}$$

which are versions of Pell's equation [10]. With these substitutions it is easy to convert between solutions of equations (1) and (2) and solutions of equations (3) and (4).

3. The Solutions. Write the equation $x^2 + axy + y^2 = b$ as $\mathbf{x}^T A\mathbf{x} = b$, where

$$\mathbf{x} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & \frac{a}{2} \\ \frac{a}{2} & 1 \end{bmatrix},$$

and apply the following theorem.

<u>Theorem 1</u>. Consider the equation $\mathbf{x}^T A\mathbf{x} = b$, where A is a square matrix, $\mathbf{x}$ is a column vector, and $b \in \mathbb{R}$. Suppose P is a matrix with

integer entries such that $P^T AP = A$. If $\mathbf{x}$ is a solution to the equation, then $P\mathbf{x}$ is also a solution. If P has an inverse, then $P^{-1}\mathbf{x}$ is also a solution.

Proof. $(P\mathbf{x})^T A(P\mathbf{x}) = \mathbf{x}^T P^T AP\mathbf{x} = \mathbf{x}^T A\mathbf{x} = b$. If P has an inverse, then $(P^{-1})^T A\ (P^{-1}) = A$, and it follows that $(P^{-1}\mathbf{x})^T A(P^{-1}\mathbf{x}) = \mathbf{x}^T (P^{-1})^T AP^{-1}\mathbf{x} = \mathbf{x}^T A\mathbf{x} = b$.

The matrix

$$P = \begin{bmatrix} -a & -1 \\ 1 & 0 \end{bmatrix}$$

satisfies the hypothesis of Theorem 1. Setting $a = -3$ and $b = -1$ gives the equation for the mother-son age problem. Since

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a solution to equation (1), all elements of the set

$$K^+ = \left\{ \mathbf{x}_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix} = P^k \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| k \in \mathbb{Z} \right\}$$

are solutions to equation (1).

Similarly, there is a set K^- of solutions corresponding to the solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

The sets K^+ and K^- give solutions on the two branches in quadrants one and three, respectively, of the hyperbola defined by equation (1). For any $k \in \mathbb{Z}$ and $\mathbf{x}_k, \mathbf{x}_{k-1} \in K^+$ (or K^-), Theorem 1 yields

$$\mathbf{x}_k = P\mathbf{x}_{k-1}. \tag{5}$$

Note that P is diagonalizable via the similarity transformation $P = QDQ^{-1}$, where

$$D = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & 0 \\ 0 & \frac{3+\sqrt{5}}{2} \end{bmatrix}$$

and

$$Q = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}.$$

Letting $\mathbf{x}_k \in K^+$ and substituting into the recursive equation (5) above gives

$$\mathbf{x}_k = P\mathbf{x}_{k-1} \;=\; P^k\mathbf{x}_0 \;=\; QD^kQ^{-1}\mathbf{x}_0.$$

Note that if ϕ is defined to be $\frac{1+\sqrt{5}}{2}$ (the Golden Ratio), then

$$\phi^2 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \phi^{-2} = \frac{3-\sqrt{5}}{2}.$$

Substitution of these values into D and Q leads to formulas for x_k and y_k:

$$x_k = \frac{\phi^{-2k} + \phi^{2k+2}}{1+\phi^2}$$

$$y_k = \frac{\phi^{-2k+2} + \phi^{2k}}{1+\phi^2}.$$

These formulas are related to the sequence of Fibonacci numbers, F_n. This sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The definition can be extended to negative integers n by $F_{-n} = (-1)^{n+1}F_n$. Binet's familiar formula [2] for the k^{th} Fibonacci number is

$$F_k \;=\; \frac{1}{\sqrt{5}}\left(\phi^k - (-1)^k\phi^{-k}\right), \; k \in \mathbb{Z}.$$

Replacing k by $2k+1$ and noting that $\dfrac{\phi}{1+\phi^2} = \dfrac{1}{\sqrt{5}}$ yields

$$\begin{aligned} F_{2k+1} &= \frac{\phi}{1+\phi^2}\left(\phi^{2k+1} - (-1)^{2k+1}\phi^{-2k-1}\right) \\ &= \frac{\phi^{-2k} + \phi^{2k+2}}{1+\phi^2} \\ &= x_k. \end{aligned}$$

The proof that $y_k = F_{2k-1}$ is similar. Thus, the solutions to equation (1) include pairs of Fibonacci numbers whose indices run over the odd integers. In fact, these are all the solutions to the equation.

The matrices

$$C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \text{ and } C^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

are used to convert between solutions to equations (1) and (3). For example,

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

is a solution to (1), and

$$\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

which is a solution to (3).

Note that solutions to (3) must consist of either two even integers or two odd integers. Thus, multiplying any solution of (3) by C^{-1} gives an *integral* solution to (1). It follows that there is a one-to-one correspondence between the solutions to equations (1) and (3).

Standard results on Diophantine equations will now reveal that the set $K = K^+ \cup K^-$ contains all solutions to equation (1). First, determine all solutions to equation (3). To describe these solutions, some new notation is introduced. If the vector

$$\begin{bmatrix} s \\ t \end{bmatrix}$$

is a solution to an equation of the form $u^2 - 5v^2 = c$, this solution can be conveniently denoted by $s + t\sqrt{5}$. Now identify the sets

$$S = \left\{ \begin{bmatrix} s \\ t \end{bmatrix} \,\middle|\, s,\ t \in \mathbb{Z} \right\} \text{ and } R = \left\{ s + t\sqrt{5} \mid s,\ t \in \mathbb{Z} \right\}.$$

Note also that the set R is closed under multiplication and is isomorphic (as a monoid) to the set

$$M = \left\{ \begin{bmatrix} s & 5t \\ t & s \end{bmatrix} \middle| \; s,\ t \in \mathbb{Z} \right\}$$

under matrix multiplication. The map $M \mapsto S$, given by

$$\begin{bmatrix} s & 5t \\ t & s \end{bmatrix} \to \begin{bmatrix} s & 5t \\ t & s \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix},$$

gives a bijection between the elements of M and S.

By [10], all solutions to equation (3) are given by the union W of the three sets

$$W_1 = \left\{ \pm(9+4\sqrt{5})^k \cdot (-1+\sqrt{5}) \mid k \in \mathbb{Z} \right\},$$

$$W_2 = \left\{ \pm(9+4\sqrt{5})^k \cdot (1+\sqrt{5}) \mid k \in \mathbb{Z} \right\},$$

$$W_3 = \left\{ \pm(9+4\sqrt{5})^k \cdot (4+2\sqrt{5}) \mid k \in \mathbb{Z} \right\}.$$

The sets W_1, W_2, and W_3 are called *classes of solutions*. The solutions $-1+\sqrt{5}$, $1+\sqrt{5}$, and $4+2\sqrt{5}$ are known as *fundamental solutions* of their respective classes, and they are found using certain numerical conditions [10]. The expression $9+4\sqrt{5}$ is a fundamental solution to the equation

$$u^2 - 5v^2 = 1.$$

Let W^+ be the subset of W whose elements begin with "+" in the three sets above, and let W^- be the subset of W whose elements begin with "−" in the three sets above. Similarly, one may speak of W_i^+ and W_i^- for $i = 1, 2, 3$.

Recall that multiplication by the matrix C gives a bijection between the solutions to equations (1) and (3). From this, it follows that the set K maps onto W. First, it is clear that K^+ maps onto W^+.

If

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} \in K^+,$$

then

$$C \cdot \begin{bmatrix} x_k \\ y_k \end{bmatrix} = C \cdot P^k \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For example,

$$C \cdot \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}^3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}.$$

Let $k \in \mathbb{Z}$ and consider $(9 + 4\sqrt{5})^k \cdot (-1 + \sqrt{5}) \in W_1^+$. One can now show that

$$C \cdot \begin{bmatrix} x_{3k} \\ y_{3k} \end{bmatrix} = (9 + 4\sqrt{5})^k \cdot (-1 + \sqrt{5}).$$

In view of the identifications between the sets S, R, and M described earlier, the previous equation reduces to the matrix equation

$$\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}^{3k} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix}^k \cdot \begin{bmatrix} -1 & 5 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which can be proved using mathematical induction. Similarly, there is a one-to-one correspondence between the solutions

$$\begin{bmatrix} x_{3k+1} \\ y_{3k+1} \end{bmatrix}$$

and W_2^+ and between the solutions

$$\begin{bmatrix} x_{3k+2} \\ y_{3k+2} \end{bmatrix}$$

and W_3^+. Thus, there is a one-to-one correspondence between K^+ and W^+. There is also a one-to-one correspondence between K^- and W^-. The main result of this paper is now established.

Theorem 2. The set K consists of all solutions to equation (1).

Similarly, all solutions to equation (2), the incorrect formulation of the mother-son problem, satisfy

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} F_{2k} \\ F_{2k-2} \end{bmatrix}.$$

To summarize, the solutions to equation (1) consist of Fibonacci numbers with consecutive *odd* indices, whereas the solutions to equation (2) consist of Fibonacci numbers with consecutive *even* indices. Hence, the coordinates of the integral points on the curves in Figure 1 are all Fibonacci numbers. Also, for all $k \in \mathbb{Z}$, the identity

$$F_k^2 - 3F_kF_{k-2} + F_{k-2}^2 = (-1)^k$$

follows immediately from the above results.

4. Further Comments and Observations. Connections between Diophantine equations (and their corresponding curves) and sequences defined by linear recurrences (such as the Fibonacci numbers) have been studied for many years. In the late 1800's, Lucas showed that if x and y are consecutive Fibonnaci numbers, then the point (x, y) lies on one of the hyperbolas $y^2 - xy - x^2 = \pm 1$, and Wasteels proved the converse in 1902 (for proofs of both results see Wasteels [11]). Lucas [7] also noted some connections between the Fibonacci numbers, Lucas numbers, and Pell's equation. (Lucas numbers, denoted by L_n, are defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$.) In the above paper, Lucas showed that $L_n^2 - 5F_n^2 = \pm 4$. More recently, Long and Jordan [6] proved the following: If (x, y) is a solution to $x^2 - 5y^2 = 4$, then $x = L_{2n}$ and $y = F_{2n}$. Correspondingly, if (x, y) is a solution to $x^2 - 5y^2 = -4$, then $x = L_{2n-1}$ and $y = F_{2n-1}$.

Bergum [1] and Horadam [3] identified some conic curves that pass through infinitely many points whose coordinates are Fibonacci numbers. Kimberling [5] then classified all hyperbolas with this property. He defined two classes of hyperbolas (here n is a positive integer):

$$p_n(x, y) = x^2 + (-1)^{n+1}L_nxy + (-1)^ny^2 + F_n^2 \quad \text{and}$$

$$q_n(x, y) = x^2 + (-1)^{n+1}L_nxy + (-1)^ny^2 - F_n^2.$$

He named these curves "Fibonacci hyperbolas" because they have the property that they pass through an infinite number of points of the form $(F_m,\ F_n)$, and he proved that these are the *only* hyperbolas with this property. Kimberling did not prove, however, that all the integer points through which these hyperbolas pass are pairs of Fibonacci numbers. We have been able to do this in the special cases of the two Fibonacci hyperbolas involved in the correct and incorrect solutions to the mother-son age problem. It is our conjecture that all the lattice points of Kimberling's hyperbolas $p_n(x, y)$ and $q_n(x, y)$ are pairs of Fibonacci numbers.

More can be said about integer solutions in the general setting. It can be shown that, if n is even, then $(1, F_{n-1})$ is a solution to the equation

$$p_n(x, y) = x^2 - L_n xy + y^2 + F_n^2 = 0.$$

In addition, this Diophantine equation can be written in quadratic form as $\mathbf{x}^T A \mathbf{x} = -F_n^2$, where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & \frac{-L_n}{2} \\ \frac{-L_n}{2} & 1 \end{bmatrix}.$$

The matrix

$$P = \begin{bmatrix} -L_n & -1 \\ 1 & 0 \end{bmatrix}$$

satisfies the hypothesis of Theorem 1, and if

$$\mathbf{x_0} = \begin{bmatrix} 1 \\ F_{n-1} \end{bmatrix},$$

then the set $\{P^k \mathbf{x_0} \mid k \in \mathbb{Z}\}$ consists of solutions to the equation.

McDaniel [8] identified some conics whose equations are satisfied by consecutive terms of certain recursively defined sequences. His Theorem 2 is a generalization of our solution to equation (2). Later, Melham [9] generalized some of McDaniel's theorems.

The proof of Theorem 1 yields no insight into how the matrix

$$P = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$$

is constructed from

$$A = \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}$$

in order to satisfy the condition $P^TAP = A$. There are other candidates for P, such as

$$\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix},$$

that would work equally well for generating all solutions to the Diophantine equation for the mother-son age problem. Also, there are many matrices that do not generate an infinite number of solutions. It is also possible to find matrices that generate an infinite number of solutions, but not *all* solutions; P^2 would be an example of such a matrix.

The matrix P that was used to obtain solutions to equation (1) was found by setting P equal to

$$\begin{bmatrix} s & t \\ 1 & 0 \end{bmatrix}$$

and solving the equation $P^TAP = A$ for s and t. This yielded two solutions: the matrix used to generate all solutions, and

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $P^2 = I$, this matrix does not generate an infinite number of solutions. The next theorem, whose proof is straightforward, puts some structure on the set of matrices P that satisfy Theorem 1.

<u>Theorem 3</u>. Let A be an invertible matrix in $M_n(\mathbb{R})$, the set of all $n \times n$ matrices over $\mathbb{R}$. Define $\mathbb{G}_A = \{P \in M_n(\mathbb{R}) \mid P^TAP = A\}$. Then $\mathbb{G}_A$ forms a group under the operation of matrix multiplication.

Other authors have used the identification of expressions of the form $s + t\sqrt{d}$ with matrices of the form

$$\begin{bmatrix} s & dt \\ t & s \end{bmatrix}$$

in problems similar to ours. See, for example, Wegener [12].

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Keith Brandt
Department of Mathematics, Computer Science, and Physics
Rockhurst University
Kansas City, MO 64110
email: keith.brandt@rockhurst.edu

John Koelzer
Department of Mathematics, Computer Science, and Physics
Rockhurst University
Kansas City, MO 64110
email: john.koelzer@rockhurst.edu