THE FUNDAMENTAL THEOREM OF ALGEBRA – PROOFS VIA "THE CREEPING LEMMA"

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Let $\mathbb{C}$ be the complex plane. A complex polynomial function is a function $P: \mathbb{C} \to \mathbb{C}$ defined by

$$P(z) = \sum_{k=0}^{n} a_{n-k} z^{n-k},$$

where $a_0, \cdots, a_n \in \mathbb{C}$. The Fundamental Theorem of Algebra states that a nonconstant complex polynomial function has a zero. Over the years, a number of proofs of this theorem have been discovered [1]. In this note, the so-called "Creeping Lemma" [3] is used to present two elementary proofs of the Fundamental Theorem. The proofs can be presented in a first analysis course (or advanced calculus course) as a nice application of diffeomorphism and the concept of least upper bound.

It is known that a complex polynomial function P is a closed function, i.e. that $P(A)$ is a closed subset of $\mathbb{C}$ when A is a closed subset of $\mathbb{C}$. The proof for non-constant P combines the Bolzano-Weierstrass property of $\mathbb{C}$ with the continuity of P and the observation that $|P(z)|$ gets arbitrarily large as $|z|$ gets arbitrarily large [2].

Here the "Creeping Lemma" is applied

(i) to provide a direct proof of The Fundamental Theorem of Algebra, and
(ii) to prove that a non-constant complex polynomial function is an open function, so $P(\mathbb{C}) = \mathbb{C}$, since $P(\mathbb{C})$ is a nonempty, closed, and open subset of the connected space $\mathbb{C}$ (another proof of the Fundamental Theorem of Algebra).

In the sequel, the set of critical points of a complex polynomial P, $\{z \in \mathbb{C} : P'(z) = 0\}$, will be denoted by M.

Theorem 1. *The Fundamental Theorem of Algebra*: A non-constant complex polynomial function has a zero.

Proof. The set of critical points, M and its image $P(M)$ are both finite. Let $p = P(a)$ with $a \notin M$. Let $\sigma: [0,1] \to \mathbb{C}$ be a curve such that $\sigma(0) = p$, $\sigma(1) = O$, and $\sigma([0,1)) \subset \mathbb{C} - P(M)$, an open connected set. It will be enough to show that there is a curve $\tilde{\sigma}: [0,1] \to \mathbb{C}$ with $\tilde{\sigma}(0) = a$ and $P \circ \tilde{\sigma} = \sigma$.

Choose a neighborhood U of a such that $P: U \to P(U)$ is a diffeomorphism with inverse g. Then, $\tilde{\sigma} = g \circ \sigma$ is defined on $[0,t]$ for some $t \in (0,1]$. Let $c = \sup\{r \in (0,1] : \tilde{\sigma}$ can be extended to a curve from $[0,r]$ into $\mathbb{C}$ such that $\tilde{\sigma}(0) = a,\ P \circ \tilde{\sigma} = \sigma\}$. Since $\sigma(c) = \lim P \circ \tilde{\sigma}(s_k)$ for a sequence $s_k \to c$ and P is a closed function, it may be assumed, without loss of generality, that $\tilde{\sigma}(s_k) \to b \in \mathbb{C}$ and $P(b) = \sigma(c)$. Suppose $c < 1$. Then, $P'(b) \neq 0$ and the point b has a neighborhood V such that $P\colon V \to P(V)$ is a diffeomorphism with inverse g_b. The curve $g_b \circ \sigma$ is defined on $(c-\epsilon, c+\epsilon) \subset (0,\ 1)$ for some $\epsilon > 0$ and ultimately $g_b \circ \sigma(s_k) = \tilde{\sigma}(s_k)$. Note that any curve $\alpha\colon (c-\epsilon, c) \to \mathbb{C}$ with $\alpha((c-\epsilon, c)) \cap V \neq \emptyset$ is uniquely determined from the condition $P \circ \alpha = \sigma$ by $\alpha = g_b \circ \sigma$. It follows that $\tilde{\sigma}$ can be extended to a curve from $[0, c+\epsilon/2]$, which is a contradiction. Hence, $c = 1$ and $P(b) = \sigma(1) = O$. The proof is complete.

Remark. At a non-critical point z of a polynomial P, the Inverse Function Theorem guarantees that there is an arbitrarily small open neighborhood of z whose image is a neighborhood of $P(z)$. A slight modification of the above proof establishes that at a critical point z of P, there is also an arbitrarily small open neighborhood of z whose image is a neighborhood of $P(z)$. Theorem 2 follows from these observations, and establishes that, for non-constant P, $P(V)$ is an open subset of $\mathbb{C}$ for each open $V \subset \mathbb{C}$, not only for $V = \mathbb{C}$.

Theorem 2. A nonconstant complex polynomial function P is an open function.

Proof. Let $V \subset \mathbb{C}$ be open and let $a \in V$. If $P'(a) \neq 0$, P is a diffeomorphism on an open neighborhood $U \subset V$ of a and $P(V)$ contains an open neighborhood $P(U)$ of $P(a)$. Suppose $P'(a) = 0$. Choose a disk $D_s(a) \subset V$ with $P^{-1}(P(a)) \cap \overline{D}_s(a) = \{a\}$ and $D^*_s(a) \subset \mathbb{C} - M$, where $D_s(p) = \{q \in \mathbb{C} : |p-q| < s\}$, $D^*_s(p) = D_s(p) - \{p\}$. Choose a number $r > 0$ satisfying the properties

(i) $2r$ is smaller than the distance from $P(a)$ to $P(\partial D_s(a))$, where $\partial D_s(a)$ is the boundary of $D_s(a)$, and
(ii) $D^*_r(P(a)) \subset \mathbb{C} - P(M)$.

Since P is not constant, choose a point $b \in D_s(a)$ such that $P(b) \in D^*_r(P(a))$. Let $q \in D^*_r(P(a))$ and let $\sigma\colon [0,1] \to D^*_r(P(a))$ be a curve with $\sigma(0) = P(b)$ and $\sigma(1) = q$. As in the proof of Theorem 1, there is a curve $\tilde{\sigma}\colon [0,1] \to \mathbb{C}$ determined by the conditions $\tilde{\sigma}(0) = b$ and $P \circ \tilde{\sigma} = \sigma$. However, $\tilde{\sigma}([0,1]) \subset D_s(a)$, since $b \in \tilde{\sigma}([0,1]) \cap D_s(a)$, and $\sigma([0,1]) \cap P(\partial D_s(a)) = \emptyset$ implies $\tilde{\sigma}([0,1]) \cap \partial D_s(a) = \emptyset$. It follows that $q = P \circ \tilde{\sigma}(1) \in P(D_s(a))$ and thus, $D_r(P(a)) \subset P(D_s(a))$. The proof is complete.

References

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