ABSOLUTE EXTREMA AND THE BAIRE CATEGORY THEOREM

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Abstract. This note ties together optimization problems from differential calculus, the Cantor function, and the Baire Category Theorem.

1. Introduction. A common misconception is that mathematics is a dry, algorithmic subject. Too often, we support this misconception in our teaching habits by providing step by step procedures which apply to typical examples of polynomial, rational, and trigonometric functions without demonstrating interesting (or pathological!) examples where our step by step procedures prove unhelpful. Mathematical beauty lies not only in our ability to develop and implement logical procedures but also in examples which test our grasp on concepts.

This note begins with a standard calculus theorem which outlines a method for finding extreme values. All too often, students are left with the impression this procedure always works or that it fails only in a rare example. Hoping to dispell this impression, we discuss classes of examples where this method is useless.

Throughout our discussions in this paper, all functions are assumed to be real-valued. A standard theorem in calculus courses (see [3] for example) is given by

Theorem 1.1. If f is a continuous function defined on a closed and bounded interval I, then f attains both its absolute maximum and absolute minimum values on I.

Finding these extrema points is central while studying optimization. The procedure generally given (see [3] for example) is outlined as follows:

Procedure 1.2 (The Closed Interval Method). If f is a continuous function defined on a closed and bounded interval $I = [a, b]$, then

Step 1. Find all critical numbers of f on I. That is, find any $c \in I$ so that $f'(c) = 0$ or $f'(c)$ fails to exist.

Step 2. Compute $f(c)$ for all critical numbers c and compute $f(a)$ and $f(b)$.

Step 3. Compare the values in step 2 to determine the absolute maximum and the absolute minimum.

A function with infinitely many critical numbers causes one to think about the definition of absolute extrema as opposed to Procedure 1.2. A typical example illustrating this phenomenon is given by the sine function defined on the real line instead of a closed and bounded interval. Here, one is able to use a modified version of Procedure 1.2 with infinitely many critical numbers due to the periodicity of the sine function.

As another example where Procedure 1.2 fails to be useful, we consider the following.

Example 1.3. Let $f(x)$ be defined on $[0,1]$ as follows:

$$f(x) = \begin{cases} x|\sin(1/x)|, & \text{if } 0 < x \leq 1; \\ 0, & \text{if } x = 0. \end{cases}$$

Since f is continuous, we could try to apply Procedure 1.2. However, we obtain a subset of critical numbers given by $c \in (0,1)$ whenever $\tan(1/c) = 1/c$. There are infinitely many solutions to this equation. Not only would one need to solve this equation but plugging in infinitely many values for comparison hardly seems feasible.

At this point, the definition of absolute extrema is of more importance than Procedure 1.2. Recalling the graph of f, we immediately see the absolute maximum of f occurring at $x = 1$ and the absolute minimum of f occurring at infinitely many points $x = 0,\ 1/(k\pi)$ for k a positive integer.

This example clearly demonstrates the limited usefulness of Procedure 1.2 while exhibiting the depth of Theorem 1.1. This paper addresses the issue as to whether there is a class of functions upon which Procedure 1.2 is useless.

2. Monotone Continuous Functions. In this section, we consider $\mathcal{MC}[a,b]$, the collection of all monotone continuous functions defined on a closed interval $[a,b]$. For $f \in \mathcal{MC}[a,b]$, we apply Theorem 1.1 to see that the extreme values are attained. Furthermore, we use the monotonicity of f to see that the extreme values are achieved at the endpoints. This direct approach is much more efficient than Procedure 1.2. In fact, under many circumstances, if one chooses to ignore the monotonicity of the function, but instead only relies on Procedure 1.2, finding the extrema of such functions can be an impossible task to complete.

The following is a well-known result of Lebesgue, see [2] for proof.

Theorem 2.1. If f is a monotone function on the interval $[a,b]$, then f is differentiable almost everywhere on $[a,b]$.

As an example of a monotone function, we remind the reader of the famous Cantor function covered in an undergraduate analysis/topology course. We briefly describe the construction of the Cantor function $\Theta(x)$ in the following example, and we refer to [2] for more details.

Example 2.2. Let $x \in [0,1]$ with the ternary expansion $\langle a_n \rangle$. Let $N := \infty$ if none of the a_n are 1, otherwise we define N to be the smallest value of n such that $a_n = 1$. Let $b_n := 1/(2a_n)$ for $n < N$ and $b_N := 1$. Then the function

$$\Theta(x) := \sum_{n=1}^{N} \frac{b_n}{2^n}$$

is a continuous nondecreasing function on the interval $[0, 1]$.

We now apply Theorem 2.1 to see that $\Theta'(x) = 0$ almost everywhere in the interval $[0, 1]$. Thus, almost all points in $[0, 1]$ are critical numbers by definition. Consequently, this will make Step 2 and Step 3 in Procedure 1.2 impossible to accomplish by any human or computer efforts.

3. Everywhere Continuous But Nowhere Differentiable Functions. Having seen a function with a list of countably many critical values (Example 1.3) and then a Cantor function (Example 2.2) with almost all of its domain being critical values, the next question should be: Is it possible to have the set of critical values equal to the domain?

In 1872, but not published until 1875, Weierstrass constructed the following everywhere continuous but nowhere differentiable function on the interval $[0, 1]$:

Example 3.1. For $x \in [0, 1]$, let

$$W(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(3^n x).$$

By Theorem 1.1, $W(x)$ will attain its extreme values on $[0, 1]$. However, since $W(x)$ is nowhere differentiable, the set of critical numbers of f will be the entire interval $[0, 1]$. Thus again, Steps 2 and 3 in Procedure 1.2 will be useless.

Finally, it is worthy to point out the relevance of this example in real analysis and topology. We remind our readers of the Baire Category Theorem.

Theorem 3.2. No nonempty open subset of a complete metric space is of first category; i.e., the countable union of nowhere dense subsets. More generally, a space X is Baire if and only if every nonempty open subset of X is of second category; i.e., it is not of first category.

The following is now a standard application of the Baire Category Theorem in many real analysis and topology textbooks, for example in [1, 2].

Proposition 3.3. In $C[a, b]$, the space of all continuous functions defined on the closed interval $[a, b]$, the set of everywhere continuous but nowhere differentiable functions on $[a, b]$ is of second category.

Hence, as suggested by Example 3.1, Procedure 1.2 will be completely useless on a space of second category.

References

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