ASYMPTOTICS ANALYSIS OF SOME BOUNDED SOLUTION TO THE GENERAL THIRD PAINLEVÉ EQUATION

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Abstract. In this paper, we study the general third Painlevé equation

$$y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

where α, β, γ, and δ are real parameters, discuss the boundedness of some solutions when $\gamma < 0$ and $\delta > 0$, and find an asymptotic representation of a group of oscillating solutions.

1. Introduction. In [1, 2, 3], some asymptotic representations and the corresponding connection formulas of the third Painlevé equations were studied. Using numerical methods, we found that the general solutions of the Painlevé equations having the form:

$$y \sim |x|^{\nu} \left(A + B\, |x|^{\tau} \cos\phi\right),$$

as $x \to \pm\infty$. For the third and the fourth general Painlevé equations, we found some relationships between ν, τ, ϕ, A, B and the parameters α, β, γ, δ in [4, 5]. In this paper, we discuss the conditions for the general third Painlevé equations to have bounded solutions and obtain some asymptotic representations of these solutions.

From this point on, we assume that $x_0 \neq 0$ and y_0 are real constants and the solution y of the third Painlevé equation

$$y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \tag{1}$$

satisfies $y(x_0) = y_0$.

Theorem 1. When $\delta > 0$, $\gamma < 0$, and $x \to \pm\infty$, the Painlevé equation (1) has a solution with the following asymptotic representation:

$$y = sgn(y_0)\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} + \frac{a}{2(-\gamma)^{\frac{1}{2}}}\,|x|^{-\frac{1}{2}}\cos\phi + \frac{b}{4(-\gamma)^{\frac{3}{4}}\delta^{\frac{1}{4}}}\,|x|^{-1}\cos^2\phi + O(|x|^{-\frac{3}{2}}), \tag{2}$$

where

$$\phi = -2(-\gamma\delta)^{\frac{1}{4}}x - \frac{1}{16(-\gamma\delta)^{\frac{3}{4}}}\left(\frac{a^2}{2} - c + \frac{b^2}{a^2} - \frac{b^2}{4a^2(-\gamma\delta)^{\frac{1}{2}}}\right)\ln|x|$$

$$+ \phi_0 + O(|x|^{-\frac{1}{2}}),$$

$$a = \left[\frac{1}{x_0}\left(\frac{\alpha^2}{-\gamma} + \frac{\beta^2}{\delta}\right) + \left(\frac{y'(x_0)}{y_0}\right)^2 - \frac{2}{x_0}\left(\alpha y_0 - \frac{\beta}{y_0}\right) - \gamma y_0^2 + \delta y_0^{-2}\right]^{\frac{1}{2}},$$

$$b = \alpha\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} + \beta\left(\frac{\delta}{-\gamma}\right)^{-\frac{1}{4}},$$

$$c = \alpha\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} - \beta\left(\frac{\delta}{-\gamma}\right)^{-\frac{1}{4}}.$$

2. The Proof of the Theorem.

Lemma 2. If $\delta > 0$ and $\gamma < 0$, and y is a solution of (1) satisfying $y(x_0) = y_0 \neq 0$, then $yy_0 > 0$ for all $|x| > |x_0|$.

Proof. If $y(x_1) = 0$ for some x_1 with $|x_1| > |x_0|$, then due to its analyticity, we can substitute the expansion

$$y(x) = C(x - x_1)^n + O((x - x_1)^{n+1}),\ C \neq 0,\ n \geq 1 \tag{3}$$

into equation (1) and obtain

$$Cn(n-1)(x - x_1)^{n-2} + O((x - x_1)^{n-1}) \tag{4}$$

$$= Cn^2(x - x_1)^{n-2} + \frac{\beta}{x} + \frac{\delta}{C}(x - x_1)^{-n} + O((x - x_1)^{n-1}).$$

Thus, $n = 1$ and $C + \frac{\delta}{C} = 0$. This contradicts the condition $\delta > 0$ and proves the lemma.

Lemma 3. If $\delta > 0$, $\gamma < 0$, and y is a solution of equation (1) satisfying $y(x_0) = y_0 \neq 0$, then there exist constants m and M such that $0 < m \leq M$ and $m \leq y(x) \leq M$ for all $|x| > |x_0|$.

Proof. Without loss of generality, we assume that $x > x_0 > 0$. We rewrite equation (1) as

$$\frac{2y'}{y}\left(\frac{y'}{y}\right)' + \frac{2y'^2}{xy^2} - \frac{2}{x}\left(\alpha + \frac{\beta}{y^2}\right)y' - 2\gamma yy' - 2\delta y^{-3}y' = 0. \tag{5}$$

Integrating (5), we get

$$\begin{aligned}&\left(\frac{y'}{y}\right)^2 + 2\int_{x_0}^{x}\frac{y'^2}{xy^2}dx - \frac{2}{x}\left(\alpha y - \frac{\beta}{y}\right) - \int_{x_0}^{x}\frac{2}{x^2}\left(\alpha y - \frac{\beta}{y}\right)dx - \gamma y^2 + \frac{\delta}{y^2}\\ &= \left(\frac{y'}{y}\right)^2(x_0) - \frac{2}{x_0}\left(\alpha y_0 - \frac{\beta}{y_0}\right) - \gamma y_0^2 + \frac{\delta}{y_0^2}.\end{aligned} \tag{6}$$

Noticing that

$$2\left|\alpha y - \frac{\beta}{y}\right| \le 2\left|\frac{\alpha}{\sqrt{-\gamma}}\sqrt{-\gamma}y\right| + 2\left|\frac{\beta}{\sqrt{\delta}y}\frac{\sqrt{\delta}}{y}\right| \le \frac{\alpha^2}{-\gamma} + \frac{\beta^2}{\delta} - \gamma y^2 + \frac{\delta}{y^2},$$

it follows from (6) that

$$\begin{aligned}\left(1 - \frac{1}{x}\right)\left(-\gamma y^2 + \frac{\delta}{y^2}\right) &\le \left(\frac{y'}{y}\right)^2 + 2\int_{x_0}^{x}\frac{y'^2}{xy^2}dx + \left(1 - \frac{1}{x}\right)\left(-\gamma y^2 + \frac{\delta}{y^2}\right)\\ &\le \int_{x_0}^{x}\frac{1}{x^2}\left(-\gamma y^2 + \frac{\delta}{y^2}\right) + C,\end{aligned} \tag{7}$$

where

$$C = \frac{1}{x_0}\left(\frac{\alpha^2}{-\gamma} + \frac{\beta^2}{\delta}\right) + \left(\frac{y'(x_0)}{y_0}\right)^2 - \frac{2}{x_0}\left(\alpha y_0 - \frac{\beta}{y_0}\right) - \gamma y_0^2 + \frac{\delta}{y_0^2}.$$

Let

$$I = \int_{x_0}^{x}\frac{1}{x^2}\left(-\gamma y^2 + \frac{\delta}{y^2}\right)dx + C.$$

Then,

$$I' = \frac{1}{x^2}\left(-\gamma y^2 + \frac{\delta}{y^2}\right) \tag{8}$$

$$\leq \frac{1}{x(x-1)}\left(1 - \frac{1}{x}\right)\left(-\gamma y^2 + \frac{\delta}{y^2}\right)$$

$$\leq \frac{I}{x(x-1)}.$$

Clearly,

$$I \leq C\frac{(x-1)x_0}{x(x_0-1)} \leq C.$$

Therefore,

$$-\gamma y^2 + \frac{\delta}{y^2} \leq \frac{Cx_0}{x_0 - 1},$$

$$\frac{\delta(x_0-1)}{Cx_0} \leq y^2 \leq \frac{Cx_0}{-\gamma(x_0-1)},$$

and

$$y'^2 \leq Cy^2 \leq \frac{C^2 x_0}{-\gamma(x_0-1)}.$$

Lemma 4. If $\delta > 0$, $\gamma < 0$, and y is a solution of equation (1) satisfying $y(x_0) = y_0 \neq 0$, then

$$y = sgn(y_0)\sqrt[4]{\frac{\delta}{-\gamma}} + |x|^{-\frac{1}{2}}\, w \tag{9}$$

and w is bounded.

Proof. Without loss of generality, we assume that $x > 0$ and $y > 0$. Let

$$y = \left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} + x^{-\frac{1}{2}} w = e^z.$$

Then,

$$z = \frac{1}{4}\ln\left(\frac{\delta}{-\gamma}\right) + x^{-\frac{1}{2}} u \quad \text{and} \quad w = \left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} x^{\frac{1}{2}} (e^{x^{-\frac{1}{2}} u} - 1).$$

Now, substituting

$$y = \left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} e^{x^{-\frac{1}{2}} u},$$

$$y' = \left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} e^{x^{-\frac{1}{2}} u}\left(-\frac{1}{2}x^{-\frac{3}{2}} u + x^{-\frac{1}{2}} u'\right),$$

and

$$\begin{aligned} y'' = & \left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} e^{x^{-\frac{1}{2}} u}\left(\frac{3}{4}x^{-\frac{5}{2}u} - x^{-\frac{3}{2}} u' + x^{-\frac{1}{2}} u''\right) \\ & + \left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} e^{x^{-\frac{1}{2}} u}\left(-\frac{1}{2}x^{-\frac{3}{2}} u + x^{-\frac{1}{2}} u'\right)^2 \end{aligned}$$

into equation (1), we obtain the equation

$$\begin{aligned} u'' + \frac{1}{4}x^{-2}u = & \ \alpha\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} x^{-\frac{1}{2}} e^{x^{-\frac{1}{2}} u} + \beta\left(\frac{\delta}{-\gamma}\right)^{-\frac{1}{4}} x^{-\frac{1}{2}} e^{-x^{-\frac{1}{2}} u} \qquad (10) \\ & - 2(-\gamma\delta)^{\frac{1}{2}} x^{\frac{1}{2}} sh(2x^{-\frac{1}{2}} u). \end{aligned}$$

We can rewrite (10) into

$$
\begin{aligned}
u'' + 4(-\gamma\delta)^{\frac{1}{2}} u = \ & \alpha\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{1}{n!} x^{-\frac{n+1}{2}} u^n \qquad (11) \\
& + \beta\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{-\frac{n+1}{2}} u^n \\
& - (-\gamma\delta)^{\frac{1}{2}} \left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} \sum_{n=1}^{\infty} \frac{2^{2n+2}}{(2n+1)!} x^{-n} u^{2n+1} - \frac{1}{4} x^{-2} u.
\end{aligned}
$$

Multiplying both sides of (11) by $2u'$ and integrating it from x_0 to x, we obtain

$$
\begin{aligned}
& u'^2 + 4(-\gamma\delta)^{\frac{1}{2}} u^2 \qquad (12) \\
= \ & C + 2\alpha\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{x^{-\frac{n+1}{2}}}{(n+1)!} u^{n+1} + \alpha\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} \sum_{n=0}^{\infty} \int_{x_0}^{x} \frac{x^{-\frac{n+3}{2}}}{n!} u^{n+1} dx \\
& + 2\beta\left(\frac{\delta}{-\gamma}\right)^{-\frac{1}{4}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{-\frac{n+1}{2}}}{(n+1)!} u^{n+1} \\
& + \beta\left(\frac{\delta}{-\gamma}\right)^{-\frac{1}{4}} \sum_{n=0}^{\infty} \int_{x_0}^{x} \frac{(-1)^n x^{-\frac{n+3}{2}}}{n!} u^{n+1} dx - \frac{1}{4} x^{-2} u^2 - \frac{1}{2} \int_{x_0}^{x} x^{-3} u^2 dx \\
& - (-\gamma\delta)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{2^{2n+3} x^{-n}}{(2n+2)!} u^{2n+2} - (-\gamma\delta)^{\frac{1}{2}} \sum_{n=1}^{\infty} \int_{x_0}^{x} \frac{2^{2n+2} x^{-n-1}}{(2n+1)!} u^{2n+2} dx,
\end{aligned}
$$

where C is a constant in terms of x_0, y_0, and $y'(x_0)$. It is clear that the last four terms in equation (12) are all less than zero. Hence,

$$u'^2 + 4(-\gamma\delta)^{\frac{1}{2}}u^2 \le 2\alpha\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}}(e^{x^{-\frac{1}{2}}u} - 1) + \alpha\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}}\int_{x_0}^{x} x^{-\frac{3}{2}}e^{x^{-\frac{1}{2}}u}dx$$

$$+ \beta\left(\frac{\delta}{-\gamma}\right)^{-\frac{1}{4}}(e^{-x^{-\frac{1}{2}}u} - 1) - \beta\left(\frac{\delta}{-\gamma}\right)^{-\frac{1}{4}}\int_{x_0}^{x} x^{-\frac{3}{2}}e^{-x^{-\frac{1}{2}}u}dx + C$$

$$\le |\alpha|\int_{x_0}^{x} x^{-\frac{3}{2}}|u|\, y dx + |\beta|\int_{x_0}^{x} x^{-\frac{3}{2}}|u|\, y^{-1}dx + C_1$$

$$\le C_2\int_{x_0}^{x} x^{-\frac{3}{2}}|u|\, dx + C_1. \tag{13}$$

Let

$$I = C_2\int_{x_0}^{x} x^{-\frac{3}{2}}|u|\, dx + C_1.$$

Then,

$$I' = x^{-\frac{3}{2}}|u| \le C_3 x^{-\frac{3}{2}}I^{\frac{1}{2}}.$$

Integrating it, we get

$$I^{\frac{1}{2}} \le C_3(x_0^{-\frac{1}{2}} - x^{-\frac{1}{2}}) + C_2.$$

Hence, u and u' are bounded, and therefore w is bounded.

Now, we continue the proof of our main theorem. We first rewrite our equation (12) as

$$u'^2 + 4(-\gamma\delta)^{\frac{1}{2}}u^2 = a^2 + 2bx^{-\frac{1}{2}}u + O(x^{-1}), \tag{14}$$

or

$$\frac{1}{u'^2 + 4(-\gamma\delta)^{\frac{1}{2}}u^2} = \frac{1}{a^2} - \frac{2}{a^4}bx^{-\frac{1}{2}}u + O(x^{-1}), \tag{15}$$

where

$$C = a^2 \quad \text{and} \quad b = \alpha\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} + \beta\left(\frac{\delta}{-\gamma}\right)^{-\frac{1}{4}}.$$

Clearly,

$$\lim_{x\to+\infty}[u'^2+4(-\gamma\delta)^{\frac{1}{2}}u^2]=C$$

and the major part of the function u should be of the form $\rho(x)\cos\phi$. Now we can let

$$u=\rho(x)\cos\phi,$$

$$u'=2(-\gamma\delta)^{\frac{1}{4}}\rho(x)\sin\phi.$$

To find the expression of ϕ, we first get its derivative:

$$\phi'=-2(-\gamma\delta)^{\frac{1}{4}}+\frac{uu''+4(-\gamma\delta)^{\frac{1}{2}}u^2}{2(-\gamma\delta)^{\frac{1}{4}}[u'^2+4(-\gamma\delta)^{\frac{1}{2}}u^2]}. \tag{16}$$

From equation (11), we can get

$$uu''+4(-\gamma\delta)^{\frac{1}{2}}u^2=bx^{-\frac{1}{2}}u+cx^{-1}u^2-\frac{8}{3}(-\gamma\delta)^{\frac{1}{2}}x^{-1}u^4+O(x^{-\frac{3}{2}}), \tag{17}$$

where

$$c=\alpha\left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}}-\beta\left(\frac{\delta}{-\gamma}\right)^{-\frac{1}{4}}.$$

We substitute (15) and (17) into equation (16) to get

$$\begin{aligned}d\phi=&-2(-\gamma\delta)^{\frac{1}{4}}dx+\frac{1}{2(-\gamma\delta)^{\frac{1}{4}}}\left[\frac{b}{a^2}x^{-\frac{1}{2}}u+\left(\frac{c}{a^2}-\frac{2b^2}{a^4}\right)x^{-1}u^2\right.\\&\left.-\frac{8(-\gamma\delta)^{\frac{1}{2}}}{3a^2}x^{-1}u^4+O(x^{-\frac{3}{2}})\right]dx,\end{aligned} \tag{18}$$

and

$$dx=\frac{-1}{2(-\gamma\delta)^{\frac{1}{4}}}\left[1+\frac{b}{4a^2(-\gamma\delta)^{\frac{1}{2}}}x^{-\frac{1}{2}}u+O(x^{-1})\right]d\phi. \tag{19}$$

Again, we substitute (19) back into (18) to get

$$
\begin{aligned}
d\phi = &-2(-\gamma\delta)^{\frac{1}{4}}dx - \frac{1}{4\,(-\gamma\delta)^{\frac{1}{2}}}\left[\frac{b}{a^2}x^{-\frac{1}{2}}u\right. \\
&+\left(\frac{c}{a^2}-\frac{2b^2}{a^4}+\frac{b^2}{4a^4(-\gamma\delta)^{\frac{1}{2}}}\right)x^{-1}u^2 \\
&\left.-\frac{8(-\gamma\delta)^{\frac{1}{2}}}{3a^2}x^{-1}u^4+O(x^{-\frac{3}{2}})\right]d\phi.
\end{aligned} \tag{20}
$$

From (14), we also can get

$$
\begin{aligned}
u &= \frac{a}{2(-\gamma\delta)^{\frac{1}{4}}}\left[1+\frac{b}{a}x^{-\frac{1}{2}}u+O(x^{-1})\right]\cos\phi \\
&= \frac{a}{2(-\gamma\delta)^{\frac{1}{4}}}\cos\phi+\frac{b}{4(-\gamma\delta)^{\frac{1}{2}}}x^{-\frac{1}{2}}\cos^2\phi+O(x^{-1}).
\end{aligned} \tag{21}
$$

Substituting (21) back into (20) again, we get

$$
\begin{aligned}
d\phi = &-2(-\gamma\delta)^{\frac{1}{4}}dx - \frac{1}{4(-\gamma\delta)^{\frac{1}{2}}}\left[\frac{b}{2a(-\gamma\delta)^{\frac{1}{4}}}x^{-\frac{1}{2}}\cos\phi\right. \\
&+\frac{1}{4(-\gamma\delta)^{\frac{1}{2}}}\left(c-\frac{b^2}{a^2}+\frac{b^2}{a^2(-\gamma\delta)^{\frac{1}{2}}}\right)x^{-1}\cos^2\phi \\
&\left.-\frac{a^2}{6(-\gamma\delta)^{\frac{1}{2}}}x^{-1}\cos^4\phi+O(x^{-\frac{3}{2}})\right]d\phi.
\end{aligned} \tag{22}
$$

Noticing the significant contribution of the terms with even power of $\cos\phi$, we rewrite (22) as

$$
\begin{aligned}
d\phi = &-2(-\gamma\delta)^{\frac{1}{4}}dx - \frac{1}{16(-\gamma\delta)^{\frac{3}{4}}}\left(\frac{a^2}{2}-c+\frac{b^2}{a^2}-\frac{b^2}{4a^2(-\gamma\delta)^{\frac{1}{2}}}\right)x^{-1}dx \\
&-\frac{b}{8a(-\gamma\delta)^{\frac{3}{4}}}x^{-\frac{1}{2}}\cos\phi d\phi+\frac{a^2}{24(-\gamma\delta)}x^{-1}\left(\cos^4-\frac{3}{8}\right)d\phi \\
&-\frac{1}{16(-\gamma\delta)}\left(c-\frac{b^2}{a^2}+\frac{b^2}{4a^2(-\gamma\delta)^{\frac{1}{2}}}\right)x^{-1}\left(\cos^2\phi-\frac{1}{2}\right)d\phi+O(x^{-\frac{3}{2}})d\phi.
\end{aligned} \tag{23}
$$

Now, integrating (23), we obtain the asymptotic expression of the function ϕ:

$$\phi = -2(-\gamma\delta)^{\frac{1}{4}}x$$
$$-\frac{1}{16(-\gamma\delta)^{\frac{3}{4}}}\left(\frac{a^2}{2} - c + \frac{b^2}{a^2} - \frac{b^2}{4a^2(-\gamma\delta)^{\frac{1}{2}}}\right)\ln x + O(x^{-\frac{1}{2}}).$$

Using the expression

$$w = \left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} x^{\frac{1}{2}}(e^{x^{-\frac{1}{2}}u} - 1)$$
$$= \left(\frac{\delta}{-\gamma}\right)^{\frac{1}{4}} u + O(x^{-\frac{1}{2}}),$$

we finish the proof of the theorem.

References

1. A. R. Its and V. Yu. Novokshenov, "The Isomonodromic Deformation Method in the Theory of Painlevé Equations," *Lecture Notes in Mathematics,* Vol. 1191, Springer-Verlag, Berlin, 1986.
2. A. V. Kitaev, "The Method of Isomonodromy Deformation and the Asymptotics of Solutions of the 'Complete' Third Painlevé Equations," *Math. USSR-sb.,* 62 (1989), 421–444.
3. A. V. Kitaev and A. H. Vartanian, "Connection Formulae for the Asymptotics of the Solutions of the Degenerate Third Painlevé Equation," *arXiv:math.,* CA/0312075. v1 3, Dec. 2003.
4. N-N. Shang and H.-Z. Qin, "Asymptotics Analysis of the General Third Painlevé Equation," Shandong University of Technology, Vol. 18, No. 1, 2004-1, 54–57.

5. H.-Z. Qin and N-N. Shang, "Asymptotics Analysis of the General Fourth Painlevé Equation," Shandong University of Technology, Vol. 18, No. 2, 2004-2, 12–15.

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