THE NUMBER OF ADMISSIBLE SEQUENCES FOR INDECOMPOSABLE SERIAL RINGS

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Abstract. We give a formula for the number of admissible sequences for indecomposable serial rings with n indecomposable projective modules whose minimum composition length is less than or equal to m. In particular, if $n = m$ is prime, we show that the number of such admissible sequences is

$$\binom{2n-1}{n} + \frac{1}{n}\left[(n-1)^2 - \binom{2n-2}{n}\right].$$

1. Introduction. To each indecomposable serial ring (with unity) R there is associated a set $\{e_1, \cdots, e_n\}$ of basic primitive idempotents and a set $\{Re_1, Re_2, \cdots, Re_n\}$ of pairwise non-isomorphic indecomposable projective left R-modules, each having a unique composition series. Denoting the composition lengths of Re_i by $c_i = c(Re_i)$, then the sequence $c_1, c_2, \cdots, c_n$ is called an *admissible sequence* for R, and satisfies the following inequalities [1]:

$$2 \leq c_l \leq c_{l-1} + 1 \quad \text{for } l = 2, \cdots, n. \tag{1.1}$$

$$c_1 \leq c_n + 1. \tag{1.2}$$

The admissible sequence for a serial ring is unique, except for cyclic permutation of the indices. We have shown that if R has a simple projective module, then the number of possible admissible sequences of length n is b_{n-1}, the $(n-1)^{st}$ Catalan number [2]. In this paper we count the number of admissible sequences for all serial rings. We adopt the convention that $\binom{s}{t} = 0$ for all integers $s < t$ and for all $s < 0$ or $t < 0$. Furthermore, we will refer to various combinatorial identities and have collected them in the appendix.

2. Counting Admissible Sequences.

Definition 2.1. Let i, j, k be positive integers and δ denote the Kronecker delta. Define $a^k_{i,j}$ as follows:

a) For all $i, k \geq 1$, $a^k_{i,1} = 0$.
b) For all $k \geq 1$ and $j \geq 2$, $a^k_{1,j} = \delta_{j,k}$.
c) For all $k \geq 1$ and $i, j \geq 2$, $a^k_{i,j} = \sum_{l \geq j-1} a^k_{i-1,l}$.

We note that each of the sums in part c) above is finite by the following lemma.

Lemma 2.2. For all $i, k \geq 1$ and for all $j \geq 0$, $a^k_{i,i+k+j} = 0$.

Proof. We induct on i. For $i = 1$, we have

$$a^k_{1,k+j+1} = \delta_{k+j+1,k} = 0 \quad \text{for all} \quad j \geq 0.$$

Let $i > 1$. Then

$$a^k_{i,i+k+j} = \sum_{l \geq (i-1)+k+j} a^k_{i-1,l}$$

and by the induction hypothesis, each term in this summation is zero.

Lemma 2.3. For all $i \geq 1$ and $j, k \geq 2$, $a^k_{i,j}$ equals the number of sequences $c_1, c_2, \cdots, c_i$ with $c_1 = k$ and $c_i = j$ that satisfy (1.1).

Proof. Fix $c_1 = k$ and induct on i. The case $i = 1$ is trivial, as $c_1 = k$ is the only such sequence and $a^k_{1,j} = \delta_{j,k}$. Let $i > 1$. Suppose $c_1, c_2, \cdots, c_i$ satisfies (1.1) with $c_i = j$. Then clearly $c_1, c_2, \cdots, c_{i-1}$ also satisfies (1.1). Conversely, if $c_1, c_2, \cdots, c_{i-1}$ satisfies (1.1), then so does $c_1, c_2, \cdots, c_i = j$ if and only if $2 \leq j \leq c_{i-1}+1$. Thus, every sequence of length i satisfying (1.1) is obtained from a sequence of length $i-1$ satisfying (1.1). In particular, the number of sequences $c_1, c_2, \cdots, c_i = j$ satisfying (1.1) is equal to the number of sequences $c_1, c_2, \cdots, c_{i-1}$ satisfying (1.1) with $c_{i-1} \geq j-1$. By induction, the latter number is $\sum_{l \geq j-1} a^k_{i-1,l}$ which by definition is $a^k_{i,j}$.

We next give a closed form of $a^k_{i,j}$.

Lemma 2.4. For all $i, j, k \geq 2$, we have

$$a^k_{i,j} = \binom{2i-j+k-3}{i-j+k-1} - \binom{2i-j+k-3}{i+k-2}.$$

Proof. We induct on i. Let $i = 2$. Then

$$a^k_{2,j} = \sum_{l \geq j-1} a^k_{1,l} = \sum_{l \geq j-1} \delta_{l,k} = \begin{cases} 1 & \text{if } j \leq k+1 \\ 0 & \text{else.} \end{cases}$$

On the other hand,

$$\binom{2i-j+k-3}{i-j+k-1} - \binom{2i-j+k-3}{i+k-2} = \binom{k-j+1}{k-j+1} - \binom{k-j+1}{k}.$$

But since $j \geq 2$, we have that $k-j+1 \leq k-1 < l$ so that $\binom{k-j+1}{k} = 0$. Furthermore, $\binom{k-j+1}{k-j+1} = 1$ if and only if $j \leq k+1$, and is zero otherwise.
Let $i > 2$. Then by the induction hypothesis,

$$a_{i,j}^k = \sum_{l \geq j-1} a_{i-1,l}^k = \sum_{l \geq j-1} \binom{2(i-1)-l+k-3}{i-1-l+k-1} - \binom{2(i-1)-l+k-3}{i-1+k-2}$$

$$= \sum_{l \geq j-1} \binom{2i+k-l-5}{i+k-l-2} - \sum_{l \geq j-1} \binom{2i+k-l-5}{i+k-3}.$$

Let A denote the first sum immediately above, and B denote the second sum. Since $i \geq 3$ we have that $2i+k-l-5 \geq i+k-l-2$. Furthermore, the last non-zero term in the summation A is when $i+k-l-2 = 0$ or $l = k+i-2$. We then apply identity (A.1) with $s = 2i+k-(j-1)-5$, $t = i+k-(j-1)-2$ and $p = l-(j-1)$ to get that

$$A = \sum_{l=j-1}^{i+k-2} \binom{2i+k-l-5}{i+k-l-2}$$

$$= \sum_{p=0}^{i+k-(j-1)-2} \binom{2i+k-5-(j-1)-p}{i+k-2-(j-1)-p}$$

$$= \binom{2i+k-(j-1)-5+1}{i+k-(j-1)-2} = \binom{2i-j+k-3}{i-j+k-1}.$$

Next, we note that the only non-zero terms in the summation B occur when $2i+k-l-5 \geq i+k-3$ or $k \leq i-2$, so that

$$B = \sum_{l=j-1}^{i-2} \binom{2i+k-l-5}{i+k-3}.$$

We first consider when $i \leq j$, in which case the summation above is empty. But if $i \leq j$, we also have that $2i - j + k - 3 < i + k - 2$ and thus, $\binom{2i-j+k-3}{i+k-2} = 0$.
Next, consider the case when $i > j$. We then apply identity (A.2) to B with $s = 2i + k - (j-1) - 5$, $t = i + k - 3$ and $p = l - (j-1)$. Thus, we have

$$
\begin{aligned}
B &= \sum_{l=j-1}^{i-2} \binom{2i+k-l-5}{i+k-3} \\
&= \sum_{p=0}^{i-j-1} \binom{2i+k-5-(j-1)-p}{i+k-3} \\
&= \binom{2i+k-(j-1)-5+1}{i+k-3+1} = \binom{2i-j+k-3}{i+k-2}.
\end{aligned}
$$

Thus, we see that $A - B$ gives the desired result.

The next lemma will be used in section 3.

Lemma 2.5. Let $i \geq 2$. Then for all r, $1 \leq r \leq i-1$,

$$\sum_{l=2}^{i} a_{i-r,i}^{l} = \binom{2i-2r-2}{i-r-1}.$$

Proof. First consider the case when $r = i - 1$. Then

$$\sum_{l=2}^{i} a_{i-r,i}^{l} = \sum_{l=2}^{i} a_{1,i}^{l} = a_{1,i}^{i} = 1.$$

Conversely,

$$\binom{2i-2r-2}{i-r-1} = \binom{2i-2(i-1)-2}{i-(i-1)-1} = \binom{0}{0} = 1.$$

Let $1 \le r \le i-2$. By Lemma 2.4 we have that

$$\sum_{l=2}^{i} a_{i-r,i}^{l} = \sum_{l=2}^{i} \left[\binom{2i-2r-i+l-3}{i-r-i+l-1} - \binom{2i-2r-i+l-3}{i-r+l-2} \right]$$

$$= \sum_{l=2}^{i} \left[\binom{i-2r+l-3}{l-r-1} - \binom{i-2r+l-3}{i-r+l-2} \right].$$

But since $r \ge 1$, we have $i-2r+l-3 \le i-r+l-4 \le i-r+l-2$ so that all of the terms above of the form $\binom{i-2r+l-3}{i-r+l-2}$ are zero. Furthermore, if $l < r+1$, then $l-r-1 < 0$ in which case all terms above of the form $\binom{i-2r+l-3}{l-r-1}$ are zero. Therefore, letting $p = l-r-1$,

$$\sum_{l=2}^{i} a_{i-r,i}^{l} = \sum_{l=r+1}^{i} \binom{i-2r+l-3}{l-r-1} = \sum_{p=0}^{i-r-1} \binom{i-r-2+p}{p}.$$

Finally, we apply identity (A.1) with $s = 2i-2r-3$ and $t = i-r-1$ to get the desired result

$$\sum_{l=2}^{i} a_{i-r,i}^{l} = \sum_{p=0}^{i-r-1} \binom{i-r-2+p}{p} = \binom{2i-2r-2}{i-r-1}.$$

For each $k \ge 2$ and $n \ge 1$, let the set of all admissible sequences $c_1, c_2, \cdots, c_n$ with $c_1 = k$ be denoted by σ_n^k. In order to satisfy (1.2) we must have that $c_n \ge c_1 - 1 = k-1$. Thus,

$$|\sigma_n^k| = \sum_{j \ge k-1} a_{n,j}^{k} = a_{n+1,k}^{k}.$$

For each $m \ge 2$ and $n \ge 1$, let $T_{n,m}$ denote the set of all admissible sequences $c_1, c_2, \cdots, c_n$ with $2 \le c_1 \le m$. Then

$$T_{n,m} = \bigcup_{l=2}^{m} \sigma_n^l$$

and

$$|T_{n,m}| = \left|\bigcup_{l=2}^{m} \sigma_n^l\right| = \sum_{l=2}^{m} a_{n+1,l}^l.$$

<u>Corollary 2.6</u>. Let $m \geq 2$ and $n \geq 1$. Then

$$|T_{n,m}| = (m-1)\binom{2n-1}{n} - \sum_{l=2}^{m} \binom{2n-1}{n+l-1}.$$

<u>Proof</u>. Applying Lemma 2.4 we get that

$$|T_{n,m}| = \sum_{l=2}^{m} a_{n+1,l}^l$$

$$= \sum_{l=2}^{m} \left[\binom{2(n+1)-l+(l-3)}{n+1-l+l-1} - \binom{2(n+1)-l+(l-3)}{n+1+(l-2)}\right]$$

$$= \sum_{l=2}^{m} \left[\binom{2n-1}{n} - \binom{2n-1}{n+l-1}\right]$$

$$= (m-1)\binom{2n-1}{n} - \sum_{l=2}^{m} \binom{2n-1}{n+l-1}.$$

<u>Corollary 2.7</u>. Let $m \geq 2$ and $n \geq 1$. Then for $m \geq n$

$$|T_{n,m}| = m\binom{2n-1}{n} - 2^{2n-2}.$$

Proof. Whenever $l > n$ we have that $n + l - 1 > 2n - 1$ so that $\binom{2n-1}{n+l-1} = 0$. We then apply identity (A.3) to Corollary 2.6 to obtain

$$|T_{n,m}|$$

$$= (m-1)\binom{2n-1}{n} - \sum_{l=2}^{m} \binom{2n-1}{n+l-1}$$

$$= (m-1)\binom{2n-1}{n} - \sum_{l=2}^{n} \binom{2n-1}{n+l-1}$$

$$= (m-1)\binom{2n-1}{n} - \left[2^{2n-2} - \binom{2n-1}{n}\right] = m\binom{2n-1}{n} - 2^{2n-2}.$$

3. Cyclic Permutations. Since the admissible sequence for a serial ring is unique only up to cyclic permutation, we next consider the equivalence classes of the elements of $T_{n,m}$ under cyclic permutation. Notice that $T_{n,m}$ does not contain all cyclic permutations of its own elements (e.g. $a = m, m+1, \cdots, m+n-1 \in T_{n,m}$ but every non-trivial cyclic permutation of a is not in $T_{n,m}$). Thus, we enlarge $T_{n,m}$ to include these elements as follows:

> For $m \geq 2$ and $n \geq 1$, let $C_n = \{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}$ denote the cyclic group of order n acting on the set $T_{n,m}$ as follows: Let $a = c_1, c_2, \cdots c_n \in T_{n,m}$, then $\alpha_0(a) = a$ and for $1 \leq i \leq n-1$, $\alpha_i(a) = c_{i+1}, \cdots c_n, c_1, \cdots c_i$. Let $S_{n,m} = \{\alpha_i(a) : a \in T_{n,m}, 0 \leq i \leq n-1\}$. Thus, $S_{n,m}$ is the set of all cyclic permutations of the elements of $T_{n,m}$ and $T_{n,m} \subset S_{n,m}$.

Regarding two admissible sequences as equivalent if one is a cyclic permutation of the other, we see that the number of equivalence classes of admissible sequences is just the number of orbits of C_n on $S_{n,m}$. We denote this number of equivalence classes by $O_{n,m}$. We first consider the size of the set $S_{n,m}$. Also, note that if $a = c_1, \cdots c_n \in S_{n,m}$, then $min\{c_1, \cdots c_n\} \leq m$.

Lemma 3.1. Let $m \geq 2$ and $n \geq 1$. Then

$$|S_{n,m} \setminus T_{n,m}| = \sum_{k=1}^{n-1} \left[k|T_{k,2}|\Big(\sum_{i=2}^{m} a^{i}_{n-k,m}\Big)\right].$$

Proof. Let $a = c_1, \cdots, c_n \in T_{n,m}$. Then all cyclic permutations of a are in $T_{n,m}$ if and only if $c_i \leq m$ for all i, $1 \leq i \leq n$. Thus, there exists a cyclic permutation of a that is not in $T_{n,m}$ if and only if there exists j with $1 \leq j \leq n-1$ such that

i) $c_j = m$.
ii) $c_{j+1} = m+1$.
iii) There exists k with $1 \leq k \leq n-j$ so that $c_i > m$ for all i with $j+1 \leq i \leq j+k$.
iv) $c_{[j+k+1]} \leq m$ (where $[p]$ denotes the least positive residue of p modulo n).

Without loss of generality, we permute a and consider $a' = \alpha_j(a)$, where

i) $a' = a_1, a_2$ with a_1 a subsequence of length k and a_2 a subsequence of length $n-k$.
ii) $a_1 = d_1, \cdots, d_k$ with all $d_i > m$ and $d_1 = m+1$.
iii) $a_2 = d_{k+1}, \cdots d_n \in T_{n-k,m}$ and $d_n = m$.
iv) $1 \leq k \leq n-1$.

Clearly, there exist k permutations of a' (and hence of a) that are not in $T_{n,m}$, namely $\alpha_i(a')$ for $0 \leq i \leq k-1$. Moreover, every element of $S_{n,m} \setminus T_{n,m}$ is of the form $\alpha_i(a')$ for some i, $0 \leq i \leq k-1$, and for some a' as above. Thus, the number of sequences in $S_{n,m} \setminus T_{n,m}$ is found by multiplying the number of sequences of the form a' by k.
The number of sequences of the form a_2 above is $\sum_{i=2}^{m} a_{n-k,m}^{i}$. Let U_k be the set of sequences of the form a_1 above. Then there is a bijection $f\colon T_{k,2} \to U_k$, where $f(d_1, \cdots d_k) = d_1 + m - 1, \cdots, d_k + m - 1$. Thus, $|U_k| = |T_{k,2}|$. Multiplying by the k cyclic permutations of a' that are not in $T_{n,m}$ and summing over all k, $1 \leq k \leq n-1$, gives the desired result.

We note that by Corollary 2.6 we have

$$|T_{k,2}| = \binom{2k-1}{k} - \binom{2k-1}{k+1} = \frac{1}{k+1}\binom{2k}{k}$$

which is the k^{th} Catalan number. Applying this to Lemma 3.1 and Corollary 2.6 we have the following corollary.

Corollary 3.2. Let $m \geq 2$ and $n \geq 1$. Then

$$
\begin{aligned}
|S_{n,m}| =& |T_{n,m}| + |S_{n,m} \setminus T_{n,m}| \\
=& (m-1)\binom{2n-1}{n} - \sum_{k=2}^{m} \binom{2n-1}{n+k-1} + \sum_{k=1}^{n-1} \left[\frac{k}{k+1} \binom{2k}{k} \left(\sum_{i=2}^{m} a^{i}_{n-k,m} \right) \right].
\end{aligned}
$$

We consider the special case when $n = m \geq 2$. Applying Corollary 3.2, Lemma 2.5, and Corollary 2.7, we have the following corollary.

Corollary 3.3. Let $n \geq 2$. Then

$$
|S_{n,n}| = n\binom{2n-1}{n} - 2^{2n-2} + \sum_{k=1}^{n-1} \frac{k}{k+1} \binom{2k}{k} \binom{2n-2k-2}{n-k-1}.
$$

Lemma 3.4. Let $n \geq 2$. Then

$$
|S_{n,n}| = (n-1)\binom{2n-1}{n}.
$$

Proof. We first note that

$$
\frac{k}{k+1}\binom{2k}{k} = \binom{2k}{k-1}
$$

and that

$$
\binom{2k}{k} - \binom{2k}{k-1} = \frac{1}{k+1}\binom{2k}{k}.
$$

We then apply (A.4) and (A.5) to Corollary 3.3 to obtain

$$
\begin{aligned}
|S_{n,n}| &= n\binom{2n-1}{n} - 2^{2n-2} + \sum_{k=1}^{n-1} \frac{k}{k+1}\binom{2k}{k}\binom{2n-2k-2}{n-k-1} \\
&= n\binom{2n-1}{n} - \sum_{k=1}^{n-1}\binom{2k}{k}\binom{2(n-k-1)}{n-k-1} + \sum_{k=1}^{n-1}\binom{2k}{k-1}\binom{2(n-k-1)}{n-k-1} \\
&= n\binom{2n-1}{n} - \sum_{k=0}^{n-1}\left[\binom{2k}{k} - \binom{2k}{k-1}\right]\binom{2(n-k-1)}{n-k-1} \\
&= n\binom{2n-1}{n} - \sum_{k=0}^{n-1}\frac{1}{k+1}\binom{2k}{k}\binom{2(n-k-1)}{n-k-1} \\
&= n\binom{2n-1}{n} - \binom{2(n-1)+1}{n-1+1} \\
&= (n-1)\binom{2n-1}{n}.
\end{aligned}
$$

4. Orbits of $\mathbf{C_n}$ on $\mathbf{S_{n,m}}$. For each $\alpha \in C_n$, let $fix(\alpha) = \{a \in S_{n,m} : \alpha(a) = a\}$. We note that for the identity α_0, $fix(\alpha_0) = S_{n,m}$. For $n \geq 1$ and $m \geq 2$, we apply Burnsides Theorem to get

$$
O_{n,m} = \frac{1}{|C_n|}\sum_{\alpha \in C_n} fix(\alpha) = \frac{1}{n}\left[|S_{n,m}| + \sum_{\alpha \in C_n \setminus \{\alpha_0\}} |fix(\alpha)|\right]. \tag{4.1}
$$

Note that for $m = 1$, we must include the number of admissible sequences of length n with $c_1 = 1$. In this case, since the rings are indecomposable with a simple projective module, no cyclic permutations are needed, and the number of such sequences is the $(n-1)^{st}$ Catalan number, $b_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$

[2]. Thus, for $n \geq 1$ and $m \geq 2$, the total number of equivalence classes of admissible sequences is

$$b_{n-1} + O_{n,m} = \frac{1}{n}\left[\binom{2n-2}{n-1} + |S_{n,m}| + \sum_{\alpha \in C_n \setminus \{\alpha_0\}} |fix(\alpha)|\right]. \qquad (4.2)$$

5. Special Case n = m is Prime. We consider the special case when $n = m$ is prime. Then for each $\alpha \in C_n \setminus \{\alpha_0\}$, the only elements of $fix(\alpha)$ are the sequences $k, k, \cdots k$ where $2 \leq k \leq n$, so that $|fix(\alpha)| = n - 1$. Applying this together with the Lemma 3.4 and (4.2), we have that the number of equivalence classes of admissible sequences is

$$b_{n-1} + O_{n,m} = \frac{1}{n}\left[\binom{2n-2}{n-1} + (n-1)\binom{2n-1}{n} + (n-1)^2\right]. \qquad (5.1)$$

Simplifying (5.1), we have the following theorem.

Theorem 5.2. Let n be prime. Then the number of equivalence classes of admissible sequences for indecomposable serial rings with n indecomposable projective modules whose minimum composition length is less than or equal to n is

$$\binom{2n-1}{n} + \frac{1}{n}\left[(n-1)^2 - \binom{2n-2}{n}\right].$$

Appendix.

$$\sum_{k=0}^{t}\binom{s-k}{t-k} = \sum_{k=0}^{t}\binom{s-t+k}{k} = \binom{s+1}{t} \text{ for } s \geq t \geq 0. \text{ ([3] pg. 7) } (A.1)$$

$$\sum_{k=0}^{s-t}\binom{s-k}{t} = \binom{s+1}{t+1} \text{ for } s \geq t \geq 0. \qquad \text{([3] pg. 7) } (A.2)$$

$$\sum_{k=1}^{s} \binom{2s-1}{s+k-1} = 2^{2s-2} \text{ for } s \geq 1. \qquad \text{([3] pg. 34)} \quad (A.3)$$

$$\sum_{k=0}^{s} \binom{2k}{k} \binom{2(s-k)}{s-k} = 2^{2s} \text{ for } s \geq 0. \qquad \text{([3] pg. 130)} \quad (A.4)$$

$$\sum_{k=0}^{s} \frac{1}{k+1} \binom{2k}{k} \binom{2(s-k)}{s-k} = \binom{2s+1}{s+1} \text{ for } s \geq 0. \qquad \text{([3] pg. 120)} \quad (A.5)$$

References

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