RINGS THAT CHARACTERIZE SOME SEPARATION NOTIONS

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Abstract. In this paper, we characterize the rings whose prime spectrum satisfy some topological notions related to the lower separation axioms. In order to do so, the notions of γ-ring, ε-ring, and ES-ring are introduced.

1. Introduction. Among the many relations that can be established between algebraic and topological structures, we consider the one that associates to any commutative ring with identity the topological space of its prime ideals endowed with the Zariski topology. This association is indeed a functorial relation from the commutative rings with identity to the topological spaces, i.e., a "dictionary" of properties between objects and morphisms of these categories. It is therefore natural to "translate" topological properties assumed in $Spec(R)$ into algebraic properties of the ring R. Some of these properties have been studied from the origins of the theory and have been included as exercises in some classic texts [2, 5] or developed partially in [1] and [8].

The aim of this paper is to characterize the rings whose spectra satisfy some topological notions like $T(\gamma)$, $T(\beta)$, $T(\varepsilon)$, T_{ES} and to show that rings exist for each one of these classes. These characterizations lead to the notion of γ-ring, ε-ring, and ES-ring.

2. Preliminaries. In this paper it is assumed that all the rings are commutative with identity. Now we present the Zariski topology defined on the set of prime ideals of a commutative ring with identity and a base for this topology. A complete exhibition of this topic can be consulted in [1, 2, 8].

Let R be a ring and $Spec(R)$ be the set of all prime ideals of R. For $E \subseteq R$ and $V(E) = \{P \in Spec(R) \mid E \subseteq P\}$, the following properties hold:

a) $V(0) = Spec(R)$, $V(1) = \emptyset$.
b) If $\{E_i\}_{i\in I}$ is a family of sets of R then $\bigcap_{i\in I} V(E_i) = V(\bigcup_{i\in I} E_i)$.
c) If E_1 and E_2 are subsets of R, then $V(E_1) \cup V(E_2) = V(\langle E_1\rangle \cap \langle E_2\rangle) = V(\langle E_1\rangle \langle E_2\rangle)$. $\langle E\rangle$ is the ideal generated by E.

The collection $\varsigma = \{V(E) \mid E \subseteq R\}$ satisfies the axioms for closed subsets of a topological space and so, defining $\tau = \{Spec(R) - V(E) \mid E \subseteq R\}$, one has that $(Spec(R), \tau)$ is a topological space, called the prime spectrum of R. τ is known as the Zariski topology. The open sets are $X - V(E)$, where $E \subseteq R$ and $X = Spec(R)$. For $f \in R$, the collection $\beta = \{X_f \mid f \in R\}$ with $X_f = X - V(f)$ is a base for the Zariski topology.

There are interesting relations between the basis elements X_f and the algebraic properties of the ring R. However, they will not be presented here and can be consulted in [1, 2, 8].

With this topology it is proven that for $P \in Spec(R)$, $\overline{\{P\}} = \{Q \in Spec(R) \mid P \subseteq Q\}$ and consequently $\{P\}' = \{Q \in Spec(R) \mid P \subset Q\}$. We recall that $(Spec(R), \subseteq)$ is an ordered set. Also, $\{P\}$ is a closed set in the topological space $Spec(R)$ if and only if P is a maximal ideal in $(Spec(R), \subseteq)$. This fact will be used repeatedly in the propositions of the following section.

<u>Definition 1</u>. Let R be a ring. A subset A of proper ideals of R is comaximal if for all $P, Q \in A$, $P \nsubseteq Q$ and $Q \nsubseteq P$, then $P + Q = R$ (P, Q are comaximal).

The following proposition will be used repeatedly in the next section. Its proof can be found in [12].

<u>Proposition 1</u>. If X is a finite poset, then there is a ring A with $Spec(A) \cong X$ (as ordered sets).

The study of the low axioms of separation began with the work of Young [14] and Yang (according to [11]). Later on, Aull and Thron [3] introduced new axioms and presented equivalent characterizations for some of them. They analyzed the inclusion relationships and observed that all of them can be described in terms of derived sets of points. Finally, McSherry [13] defined and studied the axiom T_{ES}.

For a topological space $(X, \mathcal{T})$ and $x \in X$, define the kernel of x, $\widehat{\{x\}}$, as the set

$$\left\{ y \in X \mid x \in \overline{\{y\}} \right\}.$$

The set

$$\widehat{\{x\}} \backslash \{x\}$$

is called shell of x and it will be denoted in this work by $\widetilde{\{x\}}$. The set A is called a point closure if $A = \overline{\{p\}}$ for some $p \in X$.

<u>Definition 2</u>. A topological space $(X, \mathcal{T})$ is:

a) $T(\gamma)$ if for any $x \in X$, $\{x\}'$ is the union of disjoint point closures.

b) $T(\gamma')$ if for any $x \in X$, $\{x\}'$ is the union of disjoint kernels.

c) $T(\beta)$ if for any $x \in X$, $\{x\}'$ is empty or singleton.

d) $T(\beta')$ if for any $x \in X$, $\widetilde{\{x\}}$ is empty or singleton.

e) $T(\varepsilon)$ if for any $x, y \in X$, $x \neq y$, $\{x\}' \cap \{y\}'$ is empty or singleton.

f) $T(\varepsilon')$ if for any $x, y \in X$, $x \neq y$, $\widetilde{\{x\}} \cap \widetilde{\{y\}}$ is empty or singleton.

g) T_{ES} if for any $x \in X$, $\{x\}$ is an open set or a closed set.

h) T_{YS} if for any $x, y \in X$, $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ is either $\emptyset$ or $\{x\}$ or $\{y\}$.

It is important to notice that the axioms $T(\beta)$, $T(\beta')$, $T(\varepsilon)$, and $T(\varepsilon')$ do not necessarily imply T_0. These topological notions will be considered in the following section. A detailed study of them can be found in [3, 13].

3. Results. Since it is always possible to associate the topological space $Spec(R)$ to each ring R, it is natural to consider rings that characterize certain topological properties of its spectrum [1, 4, 8]. The aim of this section is to characterize the rings whose prime spectrum satisfy the topological notions defined in the previous section. Here we will only prove the necessity of the conditions, since the sufficiency is straightforward.

Definition 3. Let R be a ring and $P \in Spec(R)$. $\min(P)$ will denote the set of the minimal elements of $\{P\}'$.

Definition 4. A ring R is a γ-ring if for all prime P non-maximal, $\min(P)$ exists and it is comaximal.

From Proposition 1, it is possible to construct γ-rings of any finite dimension. $\mathbb{Z}$ and any valuation domain whose spectrum is isomorphic to $\mathbb{Z}$ as posets are γ-rings.

Proposition 2. $Spec(R)$ is $T(\gamma)$ if and only if R is a γ-ring.

Proof. $\Rightarrow$) Suppose that $Spec(R)$ is $T(\gamma)$ and R is not a γ-ring. Then for some non-maximal prime ideal P, the set $\min(P) = \emptyset$ or $\min(P)$ is not comaximal. If $\min(P) = \emptyset$, then there are an infinite number of prime ideals P_1, P_2, P_3, $\ldots$, such that, $P \subset \cdots \subset P_3 \subset P_2 \subset P_1$. Thus, $\{P\}'$ is not the union of point closures, which is false. If the set $\min(P)$ is not comaximal, then there are different non-maximal prime ideals $T, S \in \min(P)$, such that, $T + S \neq R$. Thus, there exists a maximal ideal M such that $T \subset M$ and $S \subset M$. Therefore, $M \in \overline{\{T\}} \cap \overline{\{S\}}$. Hence, the sets $\overline{\{T\}}$ and $\overline{\{S\}}$ are not disjoint point closures, which is false.

Dually, $\max(P)$ and γ'-ring can be defined to characterize the rings whose spectra are $T(\gamma')$.

The *pm*-rings are the rings for which each prime ideal is contained in only one maximal ideal [6] and the *m*-rings are the rings where each prime ideal contains only one minimal prime ideal. The first class characterizes the normality of $Spec(R)$ and the second the axiom T_{YS} (with $\dim R \leq 1$) [4]. Examples of these rings are in [10] or can be constructed using Proposition 1.

Proposition 3. $Spec(R)$ is $T(\beta)$ if and only if R is a *pm*-ring and $\dim R \leq 1$.

Proof. $\Rightarrow$) Suppose that $Spec(R)$ is $T(\beta)$ and R is not a *pm*-ring or $\dim R \geq 2$. If R is not a *pm*-ring, then there exists a prime ideal P, such that, $P \subset S$ and $P \subset T$ with T and S two different maximal ideals. Thus, $S, T \in \{P\}'$, which is false. If $\dim R \geq 2$, there are different prime ideals P, Q, T such that $P \subset Q \subset T$. Thus, $Q, T \in \{P\}'$, which is also false.

Dually to the previous result, it is obtained that $Spec(R)$ is $T(\beta')$ if and only if R is an *m*-ring and $\dim R \leq 1$. This allows us to conclude that the topological notions T_{YS} and $T(\beta')$ coincide in $Spec(R)$. Also, notice that if $Spec(R)$ is $T(\beta)$, then $Spec(R)$ is normal.

Definition 5. Let R be a ring, M, N maximal ideals and T, S minimal prime ideals. Then, $\mathcal{P}_{\subset}(M, N) = \{P \in Spec(R) \mid P \subset M \cap N\}$ and $\mathcal{P}_{\supset}(T, S) = \{P \in Spec(R) \mid T \subset P \text{ and } S \subset P\}$.

Definition 6. A ring R is an ε-ring if for any M, N different maximal ideals and T, S different minimal prime ideals, the sets $\mathcal{P}_{\subset}(M, N)$ and $\mathcal{P}_{\supset}(T, S)$ are empty or singleton.

Notice that any ring of dimension zero, the valuation domains and the domains of dimension 1 (in particular $\mathbb{Z}$) are ε-rings. Also, from Proposition 1, the rings of this class of any finite dimension can be constructed.

Proposition 4. $Spec(R)$ is $T(\varepsilon)$ if and only if R is an ε-ring and $\dim R \leq 2$.

Proof. $\Rightarrow$) Suppose that $Spec(R)$ is $T(\varepsilon)$ and that R is not an ε-ring or $\dim R \geq 3$. If R is not an ε-ring, then there are different maximal ideals M, N or different minimal prime ideals T, S, such that, the set $\mathcal{P}_{\subset}(M, N)$ or the set $\mathcal{P}_{\supset}(T, S)$ is not empty or singleton. Then, there are different prime ideals $A, B \in \mathcal{P}_{\subset}(M, N)$ or $C, D \in \mathcal{P}_{\supset}(T, S)$. Thus, $M, N \in \{A\}' \cap \{B\}'$

or $C, D \in \{T\}' \cap \{S\}'$, which is false. If $\dim R \geq 3$, then there are different prime ideals P, Q, S, T such that $P \subset Q \subset S \subset T$. Thus, $S, T \in \{P\}' \cap \{Q\}'$, which is also false.

Notice that the notion of ε-ring is auto-dual, therefore, the properties $T(\varepsilon)$ and $T(\varepsilon')$ are equivalent in $Spec(R)$.

Definition 7. A ring R is an ES-ring if for any minimal non-maximal prime ideal M,

$$\bigcap_{\substack{P \in Spec(R) \\ P \neq M}} P \not\subseteq M.$$

Notice that any ring of dimension zero or any ring with finite spectrum (in particular, the valuation domains of finite dimension and the semi-local domains of dimension 1) is an ES-ring. Also, from Proposition 1, rings of this class of any finite dimension can be built. $\mathbb{Z}$ is not an ES-ring.

Proposition 5. $Spec(R)$ is T_{ES} if and only if R is an ES-ring and $\dim R \leq 1$.

Proof. $\Rightarrow$) Suppose that $Spec(R)$ is T_{ES} and R is not an ES-ring or $\dim R \geq 2$. If R is not an ES-ring, then there exists a minimal non-maximal prime ideal M, such that,

$$\bigcap_{\substack{P \in Spec(R) \\ P \neq M}} P \subseteq M.$$

Thus, $\{M\}$ is not an open set; since otherwise we have $\{M\} = (V(A))^c$ for some $A \subseteq R$. Then, $A \not\subseteq M$ and

$$A \not\subseteq \bigcap_{\substack{P \in Spec(R) \\ P \neq M}} P.$$

Therefore, there exists a prime ideal $P \neq M$ such that $A \not\subseteq P$. Thus, $P \in (V(A))^c = \{M\}$, which is false. But, $\{M\}$ neither is a closed set, since it is not a maximal ideal, which is false. If $\dim R \geq 2$, then there are different prime ideals Q, S, T such that $Q \not\subseteq S \not\subseteq T$. The singleton $\{S\}$ is not an open set, since otherwise we have $\{S\} = (V(A))^c$ for some $A \subseteq R$.

Then, $A \not\subseteq S$ and $A \not\subseteq Q$. Therefore, $Q \in (V(A))^c = \{S\}$, which is false. But the singleton $\{S\}$ neither is a closed set, which is also false.

4. Concluding Remarks. Notice that the definition of γ-ring, ε-ring, and ES-ring, allows us to outline new research questions about their algebraic properties. Some of the topics that could be studied are: dimension, ring of fractions, quotient ring, product ring, ring of polynomials, ring of formal power series and extensions of ring. Finally, we notice that any topological notion leads to a class of rings with special characteristics, namely, the functor *Spec* allows to define new classes of rings. Also, we can characterize well-known rings in terms of topological properties on its spectra or even find rings that are related to well-known rings. For example, a Noetherian ring R is Artinian if and only if $Spec(R)$ is discrete [2]; a ring R is a D-ring if and only if $Spec(R)$ is a T_D space [4]. The D-rings are defined as rings where every non-maximal prime ideal P is different from the intersection of the prime ideals strictly containing it, i.e., they are related to the rings of Jacobson.

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References

1. R. E. Ahumada, *El funtor Spec,* Tesis de Maestría, Universidad Nacional de Colombia, Medellín, 1983.
2. M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra,* Addison-Wesley Publishing Company, Massachusetts, 1969.
3. C. E. Aull and W. J. Thron, "Separations Axioms Between T_0 and T_1," *Indag. Math.,* 24 (1962), 26–37.
4. J. A. Ávila, "*Spec*(R) and Separation Axioms Between T_0 and T_1," Divulg. Mat. 13 (2005), 90–98.
5. N. Bourbaki, *Algèbre Commutative,* Hermann, París, 1961.
6. G. DeMarco and A. Orsatti, "Commutative Rings in Which Every Prime Ideal is Contained in a Unique Maximal," *Proc. Amer. Math. Soc.,* 30 (1971), 459–466.
7. J. Dieudonnè, *Cours de Gèomètrie Algèbrique,* I. PUF, París, 1974.

8. J. S. Fernández, *The Prime Spectrum of a Ring: A Survey,* Thesis Florida Atlantic University, Boca Ratón, 1991.

9. R. Gilmer, *Multiplicative Ideal Theory,* Marcel Dekker, New York, 1972.

10. L. Gillman, and M. Jerison, *Rings of Continuous Functions,* Van Nostrand, New Jersey, 1962.

11. J. L. Kelley, *General Topology,* Springer-Verlag, New York, 1955.

12. W. J. Lewis, "The Spectrum of a Ring as a Partially Ordered Set," *J. Algebra,* 25 (1973), 419–434.

13. D. M. McSherry, "On Separation Axioms Weaker Than T_1," *Proc. Roy. Irish Acad.,* 74A (1974), 115–118.

14. J. W. Young, "A Note on Separation Axioms and Their Applications in the Theory of a Locally Connected Topological Space," *Bull. Amer. Math. Soc.,* 49 (1943), 383–385.

15. O. Zariski and P. Samuel, *Commutative Algebra,* Van Nostrand, New York, 1958.

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