FACTORIZATION IN QUANTUM PLANES

Romain Coulibaly and Kenneth Price

Abstract. These results stem from a course on ring theory. Quantum planes are rings in two variables x and y such that $yx = qxy$ where q is a nonzero constant. When $q = 1$, a quantum plane is simply a commutative polynomial ring in two variables. Otherwise, a quantum plane is a noncommutative ring.

Our main interest is in quadratic forms belonging to a quantum plane. We provide necessary and sufficient conditions for quadratic forms to be irreducible. We find prime quadratic forms and consider more general polynomials. Every prime polynomial is irreducible and either central or a scalar multiple of x or of y. Thus, there can only be primes of degree 2 or more when q is a root of unity.

1. Introduction. Throughout F denotes a commutative field of characteristic zero. We often let F equal $\mathbb{Q}$, the rational number field, or $\mathbb{R}$, the real number field. Let $\mathbb{N}$ denote the set of nonnegative integers and let $\mathbb{N}^2$ denote the set of ordered pairs with entries from $\mathbb{N}$.

This work was motivated by an abstract algebra course taught by Kenneth Price. Many of the results were subsequently obtained by Romain Coulibaly while still an undergraduate student. Our interest stems from the course topics on principal ideal domains (PID's) and unique factorization domains (UFD's). For example, $F[x]$ is a PID. Every PID is a UFD but $F[x, y]$ is a UFD which is not a PID. A standard result, which generalizes Euclid's Lemma, is that irreducible elements of a UFD are prime [1].

Background material is covered in Section 2. Next, we begin the study of quadratic forms and introduce a quantum analog of the discriminant in Section 3. We give conditions for a quadratic form to be reducible. We provide examples of reducible quadratic forms which have two distinct factorizations into irreducible polynomials when $F = \mathbb{R}$.

Prime elements of general noncommutative rings are discussed in Section 4. Prime elements of quantum planes are covered in Section 5. Examples of prime quadratic forms are provided. We show any prime polynomial must be central. A final example is an irreducible and central polynomial which is not prime.

2. The Quantum Plane.

Notation 1. Let $q \in F \setminus \{0\}$ be arbitrary. The quantum plane, denoted $\mathcal{O}_q\left(F^2\right)$, is a ring generated by variables x, y whose product is governed by equation 1.

$$yx = qxy. \tag{1}$$

Quantum planes are well-known to noncommutative ring theorists and were first introduced as the noncommutative analog of a coordinate ring of a two dimensional space by Yuri Manin [7]. Our notation follows [2] and [5], although we have interchanged x and y.

Definition 2. Elements of $\mathcal{O}_q\left(F^2\right)$ are called polynomials. The terms of $\mathcal{O}_q\left(F^2\right)$ have the form $x^i y^j$ where i, j are nonnegative integers and $1 = x^0 y^0$.

We may denote a polynomial by $f(x, y)$ or simply by f. The set of terms is a basis of $\mathcal{O}_q\left(F^2\right)$. For each $f \in \mathcal{O}_q\left(F^2\right)$ there is a unique expression in equation 2 with $\lambda_{(i_1,i_2)} \in F$ and only finitely many $\lambda_{(i_1,i_2)} \neq 0$ for $(i_1, i_2) \in \mathbb{N}^2$.

$$f = \sum_{(i_1,i_2)\in\mathbb{N}^2} \lambda_{(i_1,i_2)} x^{i_1} y^{i_2}. \tag{2}$$

We often do not specify a value for q, but treat it like a variable which commutes with x, y, and elements of F. To simplify the product of elements of $\mathcal{O}_q\left(F^2\right)$ we put x to the left of y in each term. This can be achieved since equation 1 extends to $y^j x^i = q^{j \cdot i} x^i y^j$ for all $i, j \in \mathbb{N}$.

Definition 3. Let f be written as in equation 2.

(1) We call $\lambda_{(i_1,i_2)}$ a coefficient of f for $(i_1, i_2) \in \mathbb{N}^2$.
(2) The degree of f, denoted $\deg(f)$, is the largest value of $i_1 + i_2$ for all $(i_1, i_2) \in \mathbb{N}^2$ such that $\lambda_{(i_1,i_2)} \neq 0$.
(3) We say f is homogeneous if $i_1 + i_2 = \deg(f)$ for all $(i_1, i_2) \in \mathbb{N}^2$ with $\lambda_{(i_1,i_2)} \neq 0$.
(4) The support of f is $\left\{x^{i_1} y^{i_2} : (i_1, i_2) \in \mathbb{N}^2 \text{ and } \lambda_{(i_1,i_2)} \neq 0\right\}$.

The proof of Lemma 4 is similar to the proof of Gauss's Lemma provided in most introductory algebra texts, such as [1].

Lemma 4. Suppose $f, g \in \mathcal{O}_q\left(F^2\right)$ are such that fg is homogeneous. If $q \neq 0$, then f and g are homogeneous.

Proof. We proceed by contradiction and assume one of f and g is not homogeneous and fg is homogeneous. Write f as in equation 2 and write

g as in equation 3 with $\mu_{(j_1,j_2)} \in F$ and only finitely many $\mu_{(j_1,j_2)} \neq 0$ for $(j_1, j_2) \in \mathbb{N}^2$.

$$g = \sum_{(j_1,j_2)\in\mathbb{N}^2} \mu_{(j_1,j_2)} x^{j_1} y^{j_2}. \tag{3}$$

Let $\prec$ be the linear ordering on $\mathbb{N}^2$ given by $(a_1, a_2) \prec (b_1, b_2)$ if $(a_1 + a_2) < (b_1 + b_2)$ or $(a_1 + a_2) = (b_1 + b_2)$ and $a_1 < b_1$ for $(a_1, a_2), (b_1, b_2) \in \mathbb{N}^2$. Choose $(i_1, i_2), (j_1, j_2) \in \mathbb{N}^2$ as small as possible with respect to $\prec$ so that $\lambda_{(i_1,i_2)} \neq 0$ and $\mu_{(j_1,j_2)} \neq 0$. Then $i_1 + i_2 + j_1 + j_2 < \deg(f) + \deg(g)$ since one of f and g is not homogeneous.

The coefficient of $x^{i_1+j_1} y^{i_2+j_2}$ in fg must be zero since fg is homogeneous of degree $\deg(fg) = \deg(f) + \deg(g)$. This yields equation 4 with the sum over all $(i'_1, i'_2), (j'_1, j'_2) \in \mathbb{N}^2$ such that $i'_1 + j'_1 = i_1 + j_1$ and $i'_2 + j'_2 = i_2 + j_2$.

$$0 = \sum q^{i'_2 j'_1} \lambda_{(i'_1,i'_2)} \mu_{(j'_1,j'_2)}. \tag{4}$$

It is easy to see $(i'_1, i'_2) \preceq (i_1, i_2)$ or $(j'_1, j'_2) \preceq (j_1, j_2)$ for each summand in equation 4. By minimality

$$\lambda_{(i'_1,i'_2)} \mu_{(j'_1,j'_2)} = 0$$

unless $(i'_1, i'_2) = (i_1, i_2)$ and $(j'_1, j'_2) = (j_1, j_2)$. Now equation 4 becomes

$$q^{i_2 j_1} \lambda_{(i_1,i_2)} \mu_{(j_1,j_2)} = 0,$$

which is a contradiction since $q \neq 0$, $\lambda_{(i_1,i_2)} \neq 0$, and $\mu_{(j_1,j_2)} \neq 0$.

3. Reducible Polynomials.

Definition 5. Let $f \in \mathcal{O}_q(F^2) \setminus \{0\}$.

(1) We say f is reducible if $f = gh$ for some nonconstant $g, h \in \mathcal{O}_q(F^2)$.
(2) We say f is irreducible if it is nonconstant and not reducible.
(3) We call f a quadratic form if it can be written as in equation 5.

$$f(x, y) = ax^2 + bxy + cy^2. \tag{5}$$

(4) The quantum discriminant of f in equation 5 is $\Delta_q(f) = b^2 - 4acq$.

We use the quantum discriminant to provide a reducibility test for quadratic forms, which we state as Theorem 6.

Theorem 6. A quadratic form f is reducible if and only if $\Delta_q(f) = d^2$ for some $d \in F$.

Proof. Write f as in equation 5. We prove the result when $a \neq 0$ and $c \neq 0$. First suppose f is reducible. By Lemma 4 we may write f as in equation 6 for some $\lambda, \mu \in F$.

$$f(x, y) = a(x + \lambda y)(x + \mu y). \tag{6}$$

A straightforward check shows $\Delta_q(f) = d^2$ where $d = a(\mu - q\lambda)$. To finish the proof we note there is a factorization of f by setting $\lambda = (b + d)/(2aq)$ and $\mu = (b - d)/(2a)$ in equation 6.

Example 7. In $\mathcal{O}_q(\mathbb{R}^2)$ there are two distinct factorizations of $x^2 - y^2$ into irreducible polynomials if q is positive and different from 1.

$$\begin{aligned} x^2 - y^2 &= \left(x + \frac{\sqrt{q}}{q} y\right)(x - \sqrt{q} y) \\ &= \left(x - \frac{\sqrt{q}}{q} y\right)(x + \sqrt{q} y). \end{aligned}$$

If $q < 0$ there are two distinct factorizations of $x^2 + y^2$ in $\mathcal{O}_q(\mathbb{R}^2)$. Simply replace $x^2 - y^2$ with $x^2 + y^2$ and $\sqrt{q}$ with $\sqrt{-q}$ in the expression above.

4. Factorization in Noncommutative Rings. Throughout this section R denotes an arbitrary (noncommutative) ring with 1. We cover some basic concepts in this more general setting.

Definition 8. Let $r \in R$.

(1) We call r normal if for all $s, t \in R$ there exists $s', t' \in R$ such that $sr = rs'$ and $rt = t'r$.
(2) We call r central if for all $s \in R$ we have $rs = sr$.
(3) We say r divides $s \in R$ and write $r|s$ if there exists $t \in R$ such that $s = rt$ or $s = tr$.
(4) We call r prime if $r|st$ implies $r|s$ or $r|t$ for all $s, t \in R$.
(5) A two-sided ideal I of R is called completely prime if R/I is a domain.

Obviously every central element is normal.

Remark 9. In $\mathcal{O}_q(F^2)$ the variables x, y are normal. Prime elements of $\mathcal{O}_q(F^2)$ are irreducible. If q is a primitive n-th root of unity, then $f \in \mathcal{O}_q(F^2)$ is central if and only if the exponent of x and the exponent of y are both multiples of n for every term in the support of f. If q is not a root of unity, then the only central polynomials are scalars. These facts are easy to prove.

The proof of Lemma 10 is straightforward.

Lemma 10. Suppose $r \in R$ is normal and I is the smallest two-sided ideal containing r.

(1) $I = \{sr : s \in R\} = \{rt : t \in R\}$.
(2) I is completely prime if and only if r is prime.

We conclude this section with a brief account of factorization in noncommutative rings. Prime elements in noncommutative Noetherian rings were first defined by A. W. Chatters in [3]. Part 5 of Definition 8 reduces to Chatters' definition if r is normal and R is a Noetherian prime polynomial identity ring. This follows immediately from Lemma 10 and Jategaonkar's principal ideal theorem [6].

5. Prime Polynomials. Given a polynomial $f(x, y) \in \mathcal{O}_q\left(F^2\right)$ and scalars $\lambda, \mu \in F$ we may form a new polynomial by replacing the variables x and y by the scalar multiples of λx and μy, respectively. We denote this new polynomial by $f(\lambda x, \mu y)$.

Lemma 11. Suppose $p(x, y) \in \mathcal{O}_q\left(F^2\right)$ is prime and $x^i y^j$ is in the support of $p(x, y)$ for some $i, j \in \mathbb{N}$. Then $p(x, y) = q^i p(x/q, y) = q^j p(x, y/q)$.

Proof. The result is obvious if $p(x, y)$ is a scalar multiple of x or y, so assume otherwise. We show $p(x, y) = q^i p(x/q, y)$. The remaining equality can be proved similarly.

A straightforward check verifies the equation $yp(x/q, y) = p(x, y)\, y$. This implies $p(x/q, y)$ is a scalar multiple of $p(x, y)$, since p is prime. Thus, for some nonzero scalar λ we have $p(x, y) = \lambda p(x/q, y)$. By comparing coefficients we obtain $\lambda = q^i$.

Lemma 11 limits the number of possible prime polynomials in $\mathcal{O}_q\left(F^2\right)$. We explain how in the proof of Theorem 12.

Theorem 12. The elements x, y of $\mathcal{O}_q\left(F^2\right)$ are prime.

(1) If q is not a root of unity, then a prime polynomial must be a scalar multiple of x or a scalar multiple of y.
(2) Suppose q is a primitive n-th root of unity. A prime polynomial is either central and irreducible or a scalar multiple of x or of y.

Proof. We noted x and y are normal in Remark 9. Using part 1 of Lemma 10 we may find ring isomorphisms $\mathcal{O}_q\left(F^2\right) / \langle x \rangle \cong F[y]$ and $\mathcal{O}_q\left(F^2\right) / \langle y \rangle \cong F[x]$. Thus, x and y are prime by part 2 of Lemma 10.

Choose prime $p(x,y) \in \mathcal{O}_q(F^2)$. Then p is irreducible by Remark 9.

(1) If q is not a root of unity, then Lemma 11 implies $p(x,y) = \mu x^i y^j$ for some $\mu \in F$ and $i,j \in \mathbb{N}$. Since p is irreducible, we conclude $i=0$ and $j=1$ or $i=1$ and $j=0$.
(2) Suppose q is a primitive n-th root of unity and p is not a scalar multiple of x or of y. Since p is irreducible, it cannot be expressed as the product of a nonconstant polynomial with x or with y. There must be terms $x^i y^0$ and $x^0 y^j$ in the support p for some $i,j \in \mathbb{N}$. By applying Lemma 11 and comparing coefficients we see the exponents of x and y in every term in the support of p are multiples of n. Thus, p fits the description of central elements in Remark 9.

Theorems 13 and 14 show prime polynomials exist which are not scalar multiples of x or y, even when $q \neq 1$.

Theorem 13. Suppose p is an irreducible polynomial belonging to either $F[x]$ or $F[y]$. If p is central in $\mathcal{O}_q(F^2)$, then p is prime in $\mathcal{O}_q(F^2)$.

Proof. The proof in both cases is similar, so we only prove the result when $p \in F[x]$ is an irreducible and central polynomial in $\mathcal{O}_q(F^2)$. By Remark 9 there is a positive integer n such that q is a primitive n-th root of unity. We use results on skew polynomial rings, which can be found in [2] and [5].

Set $R = F[x]/(I \cap F[x])$. Then R is a commutative domain, which we may view as an F-subalgebra of $\mathcal{O}_q(F^2)/I$. There is an F-algebra automorphism $\alpha: F[x] \to F[x]$ such that $\alpha(x) = qx$ and $\alpha^n = id_R$. Since $\alpha(p) = p$, there is an induced F-algebra endomorphism from R to R, which we also denote by α. It is easy to show that α is an automorphism of R. We can construct a skew-polynomial ring $R[\hat{y};\alpha]$ and we will prove $\mathcal{O}_q(F^2)/I \cong R[\hat{y};\alpha]$.

There is an F-algebra homomorphism $\varphi: \mathcal{O}_q(F^2) \to R[\hat{y};\alpha]$ such that $\varphi(x) = x + I \cap R$ and $\varphi(y) = \hat{y}$. Since $\varphi(p) = 0$, there is an induced F-algebra homomorphism from $\mathcal{O}_q(F^2)/I$ to $R[\hat{y};\alpha]$, which we also denote by φ. On the other hand, there is an F-algebra homomorphism $\psi: R[\hat{y};\alpha] \to \mathcal{O}_q(F^2)/I$ such that $\psi|_R = id_R$ and $\psi(\hat{y}) = y + I$. A straightforward check shows ψ and φ are inverses. Thus, $\mathcal{O}_q(F^2)/I$ is isomorphic to $R[\hat{y};\alpha]$, which is a domain. Therefore, p is prime by Lemma 10.

Theorem 14. Let F be contained in $\mathbb{R}$ and consider a quadratic form $f = ax^2 + bxy + cy^2$ with $\Delta_q(f) < 0$. Then f is irreducible, but it may not be prime.

(1) If $q \neq \pm 1$, then f is not prime.
(2) If $q = 1$, then f is prime.

(3) If $q = -1$, then f is prime if and only if $b = 0$.

Proof. First of all, f is irreducible by Theorem 6.

(1) If $q \neq \pm 1$, then f is not prime by Theorem 12.
(2) If $q = 1$, then f is prime.
(3) Suppose $q = -1$. If $b \neq 0$, then Remark 9 implies f is not central and hence, not prime by Theorem 12. We assume $b = 0$ and show f is prime. It is enough to handle the case when $F = \mathbb{R}$, since $\mathcal{O}_q\left(F^2\right)$ is contained in $\mathcal{O}_q\left(\mathbb{R}^2\right)$. Then $f = a\big(x^2 - (\lambda y)^2\big)$, where $\lambda = \sqrt{-a^{-1}c}$.

Set $S = \mathcal{O}_{-1}\left(\mathbb{R}^2\right)/I$, where I is the principal ideal generated by f. Let $\mathbb{H}$ denote the division ring of quaternions containing i, j, k such that $i^2 = j^2 = k^2 = -1$ and $ji = -ij$, $ki = -ik$, and $kj = -jk$. There is a ring homomorphism $\phi\colon S \to \mathbb{H}\left[t\right]$ determined by $\phi\left(x + I\right) = it$ and $\phi\left(y + I\right) = \lambda^{-1}jt$.

We show that if $p \in \mathcal{O}_{-1}\left(\mathbb{R}^2\right)$ and $\phi\left(p + I\right) = 0$, then $p \in I$. We may find $p_0, p_1 \in \mathbb{R}\left[y\right]$ such that $p + I = \left(p_0\left(y\right) + p_1\left(y\right)x\right) + I$ by applying the left division algorithm in [4] to p divided by f. Equation 7 is immediate.

$$\phi\left(p + I\right) = \left(p_0\left(\lambda^{-1}jt\right)\right) + \left(tp_1\left(\lambda^{-1}jt\right)\right)i. \tag{7}$$

If $\phi\left(p + I\right) = 0$, then we use equation 7 to obtain $p_0 = p_1 = 0$ and $p \in I$. Thus, ϕ is injective. Now we can conclude that S has no zero divisors, since it is isomorphic to a subring of $\mathbb{H}\left[t\right]$. Thus, f is prime by Lemma 10.

Example 15. Theorems 13 and 14 imply $x^4 + 2$ and $x^2 - y^2$ are both prime in $\mathcal{O}_{-1}\left(\mathbb{Q}^2\right)$.

Now it is easy to see that $\mathcal{O}_{-1}\left(\mathbb{Q}^2\right)$ contains many prime polynomials, all of which are irreducible and central. We finish with an example of a polynomial in $\mathcal{O}_{-1}\left(\mathbb{Q}^2\right)$ which is not prime, even though it is irreducible and central.

Example 16. The polynomial $p = x^4 + y^4 \in \mathcal{O}_{-1}\left(\mathbb{Q}^2\right)$ is central. But p is not prime, since if $f = x^3 - x^2y + xy^2 - y^3$ and $g = x(1-x) - (1+x)y$, then p divides $fg = (1-x)p$ but p does not divide f or g. We claim p is also irreducible. By Lemma 4 it is enough to rule out the following two cases.

Case 1: $p = \left(a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3\right)\left(b_0x + b_1y\right)$.
Case 2: $p = \left(a_0x^2 + a_1xy + a_2y^2\right)\left(b_0x^2 + b_1xy + b_2y^2\right)$.

The proof is similar in each case, but the second is harder. We only provide the details in Case 2. Expanding and comparing coefficients yields equations i-v, which we prove are inconsistent if $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{Q}$.

i. $a_0 b_0 = 1$.
ii. $a_0 b_1 + a_1 b_0 = 0$.
iii. $a_0 b_2 - a_1 b_1 + a_2 b_0 = 0$.
iv. $a_1 b_2 + a_2 b_1 = 0$.
v. $a_2 b_2 = 1$.

If $a_1 = 0$ or $b_1 = 0$, then multiply iii. by $b_0 b_2$ and substitute i. and v. This gives $(b_0)^2 + (b_2)^2 = 0$ which, together with i. and v., yields a contradiction.

We are left with $a_i \neq 0$ and $b_i \neq 0$ for $0 \leq i \leq 2$. Multiply ii. by b_0 and substitute i. to obtain equation 8.

$$b_1 + a_1 (b_0)^2 = 0. \tag{8}$$

A similar calculation, using iv. and v., gives $b_1 + a_1 (b_2)^2 = 0$. Thus, $(b_0)^2 = (b_2)^2$ and $b_0 = s b_2$ for $s = \pm 1$. Substitute $b_0 = s b_2$ and $b_2 = s b_0$ into iii. and use i. and v. to obtain $2s = a_1 b_1$. If we multiply equation 8 by a_1 and substitute $2s = a_1 b_1$ we find $(a_1 b_0)^2 = -2s$, which is a contradiction.

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Romain Coulibaly
Wylie Hall Room 105
100 S. Woodlawn
Bloomington, IN 47405
email: rcouliba@indiana.edu

Kenneth Price
Department of Mathematics
University of Wisconsin - Oshkosh
800 Algoma Boulevard
Oshkosh, WI 54901
email: pricek@uwosh.edu