GROUP OF ISOMETRIES OF THE CC-PLANE

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Abstract. In this work, it is shown that the group of isometries of the plane with respect to the Chinese Checkers metric is the semi-direct product of the Dihedral group D_8 and $T(2)$, where D_8 is the (Euclidean) symmetry group of the regular octagon and $T(2)$ is the group of all translations of the plane. Furthermore, some properties of the CC-plane are studied and the area formula for a triangle is given.

1. Introduction. One of the basic problems in geometric investigations for a given space S with a metric d is to describe the group G of isometries. If S is the Euclidean plane with the usual metric, then it is well-known that G consists of all translations, rotations, reflections and glide reflections of the plane. It is known that for the Euclidean plane, $G = E(2)$ is the semi-direct product of its two subgroups $O(2)$ (the orthogonal group) and $T(2)$, where $O(2)$ is the symmetry group of the unit circle and $T(2)$ (the translation group) consists of all translations of the plane [2, 6, 8]. The group of isometries of the taxicab plane has been given in [7].

For the general problem stated above we use the analytical plane $\mathbb{R}^2$ endowed with the Chinese Checker Metric d_c defined by

$$d_c(X,Y) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + (\sqrt{2} - 1)\min\{|x_1 - x_2|, |y_1 - y_2|\},$$

where $X = (x_1, y_1)$ and $Y = (x_2, y_2)$. We use $\mathbb{R}_c^2 = (\mathbb{R}^2, d_c)$ for the Chinese Checkers plane (CC-plane). E. F. Krause [4] asked the question of how to develop a metric which would be similar to the movement made by playing Chinese Checkers. Later, the above metric was developed by G. Chen [1].

According to the definition of d_c-metric, the shortest path between the points A and B is the union of a vertical or a horizontal line segment and a line segment with the slope 1 or -1, as shown in Figure 1. Thus, the shortest distance d_c from A to B is the sum of the Euclidean lengths of these two line segments.

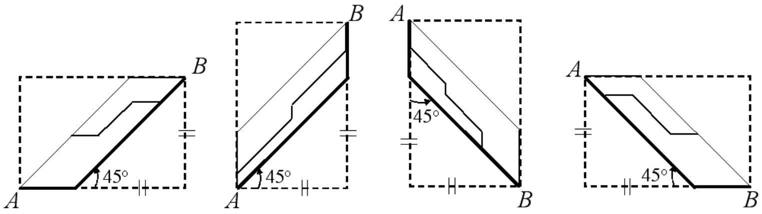

Figure 1: Some of ways from A to B .

If the slope of the segment $\overline{AB}$ is -1, 0, 1, ∞, then the d_c-distance is equal to the Euclidean distance between A and B. For the sake of brevity of notation we denote the slope of the vertical lines by ∞.

The unit circle in $\mathbb{R}_c^2$ is the set of points (x, y) in the plane which satisfy the equation

$$\max\{|x|, |y|\} + (\sqrt{2} - 1)\min\{|x|, |y|\} = 1.$$

This is an octagon with vertices

$$A_1\,(1,0)\,,\ A_2\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ A_3\,(0,1)\,,\ A_4\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$A_5\,(-1,0)\,,\ A_6\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right),\ A_7\,(0,-1)\,,\ A_8\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

as shown in Figure 2.

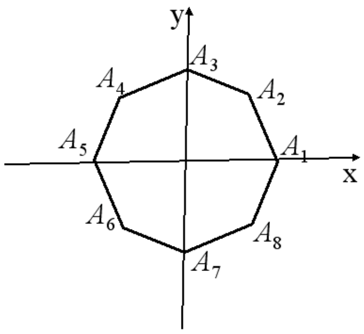

Figure 2: Graph of the CC-Unit Circle

In the remaining part of this work, we will study the isometries of $\mathbb{R}_c^2$, determine its group of isometries, and give some properties of $\mathbb{R}_c^2$.

2. Isometries of the CC-Plane ($\mathbb{R}_c^2$). Since an isometry of a plane is defined to be a transformation which preserves the distances in the plane, an isometry of $\mathbb{R}_c^2$ is therefore an isometry of the real plane with respect to the d_c metric.

Proposition 1. Every Euclidean translation is an isometry of $\mathbb{R}_c^2$.

Proof. Let $T_A : \mathbb{R}_c^2 \to \mathbb{R}_c^2$ such that $T_A(X) = A + X$ be the translation in the real plane $\mathbb{R}^2$, where $A = (a_1, a_2)$ and $X = (x_1, y_1) \in \mathbb{R}_c^2$. For $X = (x_1, y_1)$ and $Y = (x_2, y_2) \in \mathbb{R}_c^2$, we have

$$d_c\left(T_A(X), T_A(Y)\right) = \max\left\{\left|(a_1 + x_1) - (a_1 + x_2)\right|, \left|(a_2 + y_1) - (a_2 + y_2)\right|\right\}$$

$$+ \left(\sqrt{2} - 1\right) \min\left\{\left|(a_1 + x_1) - (a_1 + x_2)\right|, \left|(a_2 + y_1) - (a_2 + y_2)\right|\right\}$$

$$= \max\left\{\left|x_1 - x_2\right|, \left|y_1 - y_2\right|\right\} + \left(\sqrt{2} - 1\right) \min\left\{\left|x_1 - x_2\right|, \left|y_1 - y_2\right|\right\}$$

$$= d_c\left(X, Y\right).$$

That is, T_A is an isometry.

Since CC-plane geometry is the study of Euclidean points, lines and angles in $\mathbb{R}_c^2$, we use the following definition and lemma to find the reflections.

Definition. Let P and l be a point and a line in $\mathbb{R}_c^2$, and let Q denote the point on l such that PQ is perpendicular to l. If P' is a point on the opposite side of the line l with respect to P such that $d_c(P,Q) = d_c(P',Q)$, then P' is called the reflection of P.

Notice that it is enough to consider the lines passing through the origin as axes of reflections because of Proposition 1.

The following lemma will be useful in determining reflections in $\mathbb{R}_c^2$.

Lemma 2. Let l be the line through the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in the analytical plane and d_E denote the Euclidean metric. If l has slope m, then

$$d_c(A,B) = \frac{M}{\sqrt{m^2+1}} d_E(A,B), \text{ where } M = \begin{cases} 1 + (\sqrt{2}-1)\,|m| & \text{if } |m| \leq 1 \\ |m| + \sqrt{2} - 1 & \text{if } |m| \geq 1. \end{cases}$$

Proof. If l is parallel to the x-axis or y-axis, then $m = 0$ and $M/\sqrt{m^2+1} = 1$ or $m \to \infty$ and $\lim_{m\to\infty}(M/\sqrt{m^2+1}) = 1$. Then, $d_c(A,B) = d_E(A,B)$ in both of the cases above. If l is not parallel to the x-axis and y-axis, then $x_1 \neq x_2$ and $y_1 \neq y_2$, $m = (y_1 - y_2)/(x_1 - x_2)$, where m is the slope of l, and

$$d_c(A,B) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + (\sqrt{2}-1)\min\{|x_1 - x_2|, |y_1 - y_2|\}$$

$$= \begin{cases} |x_1 - x_2|\left(1 + (\sqrt{2}-1)\,|m|\right) & \text{if } |m| \leq 1 \\ |x_1 - x_2|\left(|m| + \sqrt{2} - 1\right) & \text{if } |m| \geq 1. \end{cases}$$

Similarly,

$$d_E(A,B) = |x_1 - x_2|\sqrt{1+m^2} \text{ for all } m \in \mathbb{R}$$

and consequently the given equality is valid.

The above proposition says that d_c-distance along any line is some positive constant multiple of Euclidean distance along the same line. Let $\rho(m)$ denote this constant, that is, $\rho(m) = M/\sqrt{m^2+1}$. For the graph of ρ see Figure 3.

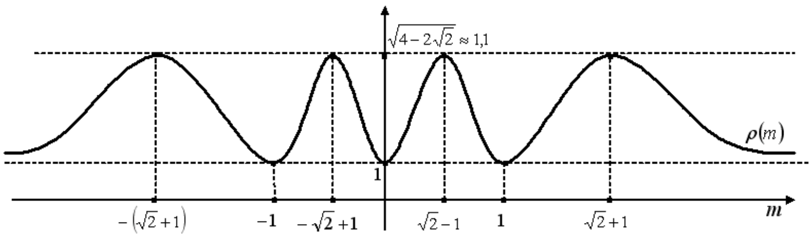

Figure 3 : Graph of ρ

Furthermore, one can immediately obtain the following corollaries.

Corollary 3. If A, B, and X are any three collinear points in $\mathbb{R}^2$, then $d_E(X, A) = d_E(X, B)$ if and only if $d_c(X, A) = d_c(X, B)$.

Corollary 4. If A, B, and X are any three distinct collinear points in the real plane, then

$$d_c(X, A)/d_c(X, B) = d_E(X, A)/d_E(X, B).$$

That is, the ratios of the Euclidean and d_c-distances along a line are the same.

Notice that the latter corollary gives us the validity of the Theorems of Menelaus and Ceva in $\mathbb{R}^2_c$. The following proposition determines the reflections which preserves distance in $\mathbb{R}^2_c$.

Proposition 5. A reflection about the line $y = mx$ in $\mathbb{R}^2_c$ is an isometry if and only if

$$m \in \left\{0, \pm 1, \pm(\sqrt{2} - 1), \pm(\sqrt{2} + 1), \infty\right\}.$$

Proof. Consider the Euclidean reflection φ about the line $y = mx$,

$$\varphi(P) = \varphi(x, y) = P' = (x', y') = \left(\frac{(1 - m^2)x + 2my}{1 + m^2}, \frac{2mx + (-1 + m^2)y}{1 + m^2}\right).$$

If $Q = \overline{PP'} \cap \{(x,y) : y = mx\}$, then $d_E(P,Q) = d_E(P',Q)$ implies $d_c(P,Q) = d_c(P',Q)$ by Corollary 3. That is, P' is a d_c-reflection of P. Using Proposition 1, one can say that φ is an isometry of $\mathbb{R}^2_c$ if and only if $d_c(O,P) = d_c(O,P')$. We claim that

$$d_c(O,P) = d_c(O,P') \text{ if and only if } m \in \left\{0, \pm 1, \pm(\sqrt{2}-1), \pm(\sqrt{2}+1), \infty\right\}.$$

If $d_c(O,P) = d_c(O,P')$, then $\rho(m_1) = \rho(m_2) = k$, where k is a constant and m_1 and m_2 are the slopes of the lines OP and OP', respectively. Furthermore, if P is on the line $y = mx$, then $P = P'$, $m_1 = m_2 = m$, $\rho(m) = k$, and the derivative $\rho'(m) = 0$ for $k \neq 1$. Thus,

$$(1+m^2)^{3/2}.\rho'(m) = \begin{cases} -1-(\sqrt{2}-1)m & \text{if } -\infty < m < -1 \\ -m-\sqrt{2}+1 & \text{if } -1 < m < 0 \\ -m+\sqrt{2}-1 & \text{if } 0 < m < 1 \\ 1-(\sqrt{2}-1)m & \text{if } 1 < m < \infty \end{cases}$$

and $\rho'(m) = 0$ gives us

$$\begin{array}{ll} m = -(\sqrt{2}+1) = tg(5\pi/8) & \text{if } -\infty < m < -1 \\ m = -(\sqrt{2}-1) = tg(7\pi/8) & \text{if } -1 < m < 0 \\ m = \sqrt{2}-1 = tg(\pi/8) & \text{if } 0 < m < 1 \\ m = \sqrt{2}+1 = tg(3\pi/8) & \text{if } 1 < m < \infty. \end{array}$$

In particular, if $\rho(m) = k = 1$, then $m \in \{0, \pm 1, \infty\}$ as seen in Figure 3.

Conversely, if m_1 and m_2 are the slopes of the lines OP and OP', respectively, where $P' = \varphi(P)$, then

$$\begin{aligned} d_E(O,P) &= d_E(O,P') \\ d_c(O,P) &= \rho(m_1)d_E(O,P) \\ d_c(O,P') &= \rho(m_2)d_E(O,P') = \rho(m_2)d_E(O,P). \end{aligned}$$

Now it is easy to check that $\rho(m_1) = \rho(m_2)$ for all of the possible cases in the following table, which implies $d_c(O, P) = d_c(O, P')$.

m	0	$\sqrt{2}-1$	1	$\sqrt{2}+1$
m_2	$-m_1$	$\frac{1-m_1}{1+m_1}$	$\frac{1}{m_1}$	$\frac{m_1+1}{m_1-1}$

m	∞	$-\sqrt{2}-1$	-1	$-\sqrt{2}+1$
m_2	$-m_1$	$\frac{1-m_1}{1+m_1}$	$\frac{1}{m_1}$	$\frac{m_1+1}{m_1-1}$

The above proposition shows that not all reflections preserve d_c-distances. The set of isometric reflections S_c consist of the eight Euclidean reflections about the lines in

$$\left\{x = 0,\ y = 0,\ y = \mp x,\ y = \mp\left(\sqrt{2}-1\right)x,\ y = \mp\left(\sqrt{2}+1\right)x\right\}.$$

Proposition 6. There are only eight Euclidean rotations that preserve d_c-distances. In other words, the set of isometric rotations in $\mathbb{R}_c^2$ is

$$R_c = \left\{r_\theta \mid \theta = k\frac{\pi}{4}, k = 0, 1, \dots, 7\right\}.$$

Proof. In order to find the isometric rotations in $\mathbb{R}_c^2$, it is sufficient to determine the rotations which preserve the lengths of the sides of the d_c-unit circle. Consider the points $A_1 = (1, 0)$ and $A_2 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ on the unit circle of $\mathbb{R}_c^2$. Rotating A_1 and A_2 by an angle θ, we get

$$\begin{aligned} r_\theta(A_1) &= (\cos\theta, \sin\theta) \\ r_\theta(A_2) &= (1/\sqrt{2})(\cos\theta - \sin\theta, \sin\theta + \cos\theta). \end{aligned}$$

Clearly, $d_c(A_1, A_2) = 2\sqrt{2} - 2$. If r_θ preserves d_c-distance, we must look for θ which implies $d_c(r_\theta(A_1), r_\theta(A_2)) = 2\sqrt{2} - 2$. Thus,

$$\begin{aligned} & d_c(r_\theta(A_1), r_\theta(A_2)) \\ & = \max\left\{\left|\tfrac{2-\sqrt{2}}{2}\cos\theta + \tfrac{\sqrt{2}}{2}\sin\theta\right|, \left|\tfrac{2-\sqrt{2}}{2}\sin\theta - \tfrac{\sqrt{2}}{2}\cos\theta\right|\right\} \\ & + (\sqrt{2}-1)\min\left\{\left|\tfrac{2-\sqrt{2}}{2}\cos\theta + \tfrac{\sqrt{2}}{2}\sin\theta\right|\ ,\ \left|\tfrac{2-\sqrt{2}}{2}\sin\theta - \tfrac{\sqrt{2}}{2}\cos\theta\right|\right\} \\ & = 2\sqrt{2} - 2. \end{aligned}$$

Let $\alpha = \frac{2-\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta$ and $\beta = \frac{2-\sqrt{2}}{2}\sin\theta - \frac{\sqrt{2}}{2}\cos\theta$. Now, consider the following cases.

i) Let $|\alpha| \geq |\beta|$.

If $\alpha \geq 0$ and $\beta \leq 0$, then $\cos\theta + \sin\theta = \sqrt{2}$ which implies $\theta = \pi/4$.
If $\alpha \geq 0$ and $\beta \geq 0$, then $\sin\theta = 1$ which implies $\theta = \pi/2$.
If $\alpha \leq 0$ and $\beta \geq 0$, then $\cos\theta + \sin\theta = \sqrt{2}$ which implies $\theta = 5\pi/4$.
If $\alpha \leq 0$ and $\beta \leq 0$, then $\sin\theta = -1$ which implies $\theta = 3\pi/2$.

ii) Let $|\alpha| < |\beta|$.

Similar to case (i), we gets $\theta = 3\pi/4$, $\theta = 0$, $\theta = \pi$, and $\theta = 7\pi/4$.

From (i) and (ii), $\theta \in \{0,\ \pi/4,\ \pi/2,\ 3\pi/4,\ \pi,\ 5\pi/4,\ 3\pi/2,\ 7\pi/4\}$. Finally, let m and m' denote the slopes of the lines OX and OX', respectively, where $X' = r_\theta(X)$. As in Proposition 5, it can easily be check that $\rho(m) = \rho(m')$ for all of the possible cases in the following table, which implies $d_c(O,X) = d_c(O,X')$. That is, every $r_\theta \in R_c$ preserves all d_c-distances by Proposition 1.

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$
m'	m	$\frac{1+m}{1-m}$	$-\frac{1}{m}$	$\frac{m-1}{m+1}$

θ	π	$5\pi/4$	$3\pi/2$	$7\pi/4$
m'	m	$\frac{1+m}{1-m}$	$-\frac{1}{m}$	$\frac{m-1}{m+1}$

Thus, we have the orthogonal group, consisting of eight reflections and eight rotations

$$O_c(2) = R_c \cup S_c,$$

which gives us the *Dihedral group* D_8, that is, the Euclidean symmetry group of the regular octagon. Now, let us show that all isometries of $\mathbb{R}_c^2$ are in $T(2).O_c(2)$.

Definition. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two points in $\mathbb{R}_c^2$. The minimum distance set of A and B is defined by

$$\{X \mid d_c(A,X) + d_c(B,X) = d_c(A,B)\}$$

and denoted by $\widehat{AB}$ (Fig. 4).

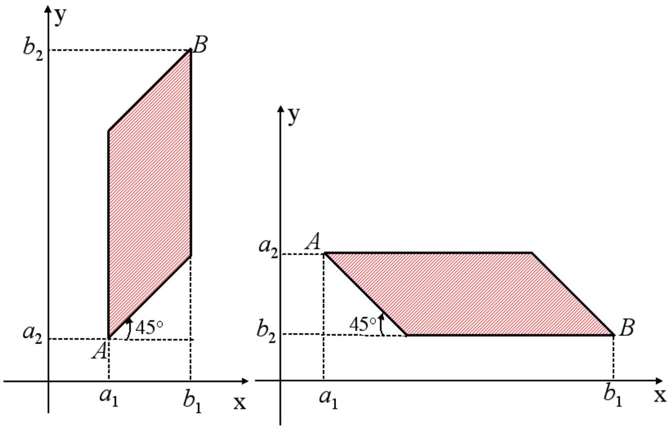

Figure 4

Let m_{AB} denote the slope of the line through the points A and B. If $m_{AB} = 0, \mp 1$, or the line is vertical, the set $\widehat{AB}$ is the line segment joining A and B, that is, $\widehat{AB} = \overline{AB}$. We call $\widehat{AB}$ the standard parallelogram with diagonal $\overline{AB}$.

<u>Proposition 7</u>. Let $\phi\colon \mathbb{R}^2_c \to \mathbb{R}^2_c$ be an isometry and let $\widehat{AB}$ be the standard parallelogram. Then

$$\phi(\widehat{AB}) = \widehat{\phi(A)\phi(B)}.$$

<u>Proof</u>. Let $Y \in \phi(\widehat{AB})$. Then the following statements are equivalent.

$Y \in \phi(\widehat{AB})$

there exists an $X \in \widehat{AB}$ such that $Y = \phi(X)$

$d_c(A, X) + d_c(X, B) = d_c(A, B)$

$d_c(\phi(A), \phi(X)) + d_c(\phi(X), \phi(B)) = d_c(\phi(A), \phi(B))$

$Y = \phi(X) \in \widehat{\phi(A)\phi(B)}$.

Corollary 8. Let $\phi\colon \mathbb{R}_c^2 \to \mathbb{R}_c^2$ be an isometry and let $\widehat{AB}$ be the standard parallelogram. Then ϕ maps vertices to vertices and preserves the lengths of sides of $\widehat{AB}$.

Proposition 9. Let $f\colon \mathbb{R}_c^2 \to \mathbb{R}_c^2$ be an isometry such that $f(O) = O$. Then $f \in R_c$ or $f \in S_c$.

Proof. Let

$$A_1 = (1, 0),\ A_2 = (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}})\ , D = (1 + \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}})$$

and consider the standard parallelogram $\widehat{OD}$.

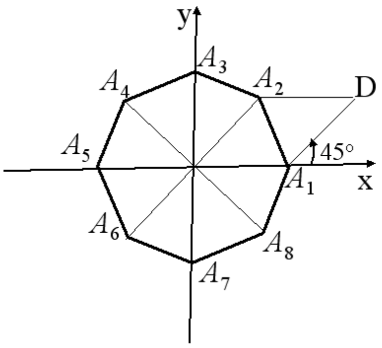

Figure 5

It is clear from Figure 5 that $f(A_1) \in \overline{A_i A_{i+1}}$. Since f is an isometry by Proposition 6, $f(A_1)$ and $f(A_2)$ must be the vertices of the standard parallelogram $\widehat{Of(D)}$. Also, when the slope of the parallel sides of a standard parallelogram is 0 or ∞, the slope of the other parallel sides is 1 or -1. Therefore, if $f(A_1) \in \overline{A_i A_{i+1}}$, then $f(A_1) = A_i$ or $f(A_1) = A_{i+1}$. Similarly, $f(A_2) = A_i$ or $f(A_2) = A_{i+1}$.

Case 1: If $f(A_1) = A_1$, then $f(A_2) = A_2$ or $f(A_2) = A_8$.

Subcase 1.1: If $f(A_2) = A_2$, then f is a rotation with $\theta = 0$.

Subcase 1.2: If $f(A_2) = A_8$, then f is a reflection about the line $y = 0$.

Case 2: If $f(A_1) = A_2$, then $f(A_2) = A_1$ or $f(A_2) = A_3$.

Subcase 2.1: If $f(A_2) = A_1$, then f is a reflection about the line $y = (\sqrt{2} - 1)x$.

Subcase 2.2: If $f(A_2) = A_3$, then f is a rotation with $\theta = \pi/4$.

Case 3: If $f(A_1) = A_3$, then $f(A_2) = A_2$ or $f(A_2) = A_4$.

Subcase 3.1: If $f(A_2) = A_2$, then f is a reflection about the line $y = x$.

Subcase 3.2: If $f(A_2) = A_4$, then f is a rotation with $\theta = \pi/2$.

Case 4: If $f(A_1) = A_4$, then $f(A_2) = A_5$ or $f(A_2) = A_3$.

Subcase 4.1: If $f(A_2) = A_5$, then f is a rotation with $\theta = 3\pi/4$.

Subcase 4.2: If $f(A_2) = A_3$, then f is a reflection about the line $y = (\sqrt{2} + 1)x$.

Case 5: If $f(A_1) = A_5$, then $f(A_2) = A_4$ or $f(A_2) = A_6$.

Subcase 5.1: If $f(A_2) = A_4$, then f is a reflection about the line $x = 0$.

Subcase 5.2: If $f(A_2) = A_6$, then f is a rotation with $\theta = \pi$.

Case 6: If $f(A_1) = A_6$, then $f(A_2) = A_5$ or $f(A_2) = A_7$.

Subcase 6.1: If $f(A_2) = A_5$, then f is a reflection about the line $y{=}-(\sqrt{2}{+}1)x$.

Subcase 6.2: If $f(A_2) = A_7$, then f is a rotation with $\theta = 5\pi/4$.

Case 7: If $f(A_1) = A_7$, then $f(A_2) = A_6$ or $f(A_2) = A_8$.

Subcase 7.1: If $f(A_2) = A_6$, then f is a reflection about the line $y = -x$.

Subcase 7.2: If $f(A_2) = A_8$, then f is a rotation with $\theta = 3\pi/2$.

Case 8: If $f(A_1) = A_8$, then $f(A_2) = A_1$ or $f(A_2) = A_7$.

Subcase 8.1: If $f(A_2) = A_1$, then f is a rotation with $\theta = 7\pi/4$.

Subcase 8.2: If $f(A_2) = A_7$, then f is a reflection about the line $y = (1 - \sqrt{2})x$.

Theorem 10. Let $f\colon \mathbb{R}^2_c \to \mathbb{R}^2_c$ be an isometry. Then there exists a unique $T_A \in T(2)$ and $g \in O_c(2)$ such that $f = T_A \circ g$.

Proof. Let $f(O) = A$ where $A = (a_1, a_2)$. Define $g = T_{-A} \circ f$. We know that g is an isometry and $g(O) = O$. Thus, $g \in O_c(2)$ and $f = T_A \circ g$ by Proposition 9. The proof of uniqueness is trivial.

3. Area Formula For CC-Triangles. The area of a triangle in the Euclidean plane can be computed by the formula $A = (1/2)bh$, which is not, in general, valid in $\mathbb{R}^2_c$. Area formulas of a taxicab triangle are given in [3] and [5]. If one knows the d_c-lengths b_c and h_c of the base and the corresponding altitude, respectively, of a triangle in $\mathbb{R}^2_c$, how can its area be computed? The following theorem answers the question and gives the Euclidean area of a triangle in terms of d_c-distances.

Theorem 11. Let b_c and h_c denote the d_c-lengths of a given side (base) and the corresponding altitude, respectively, of a triangle in $\mathbb{R}_c^2$. If the slope of base is m, then the area of the triangle can be computed by

$$A = \frac{1+m^2}{2M^2} b_c h_c, \text{ where } M = \begin{cases} 1 + (\sqrt{2}-1)\,|m| & \text{if} \quad |m| \leq 1 \\ |m| + \sqrt{2} - 1 & \text{if} \quad |m| \geq 1. \end{cases}$$

Proof. If b, h, and b_c, h_c are the Euclidean and d_c-lengths of the base and the corresponding altitude of a triangle, and if the slope of the base is m, then the slope of altitude is $(-1/m)$ and

$$b = \frac{1}{\rho(m)} b_c = \frac{(1+m^2)^{1/2}}{M} b_c, \quad h = \frac{1}{\rho(-m^{-1})} h_c = \frac{(1+m^2)^{1/2}}{M} h_c$$

by Lemma 2. Using these values of b and h in the area formula one obtains

$$S = \frac{1+m^2}{2M^2} b_c h_c.$$

Clearly, the above result can be easily extended to the areas of polygons.

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