EMPIRICAL RESULTS ON OPERATIONS OF BIPOLAR FUZZY GRAPHS WITH THEIR DEGREE

SOUMITRA POULIK AND GANESH GHORAI

Abstract. Theoretical concepts of crisp graphs are highly utilized in computer science and applications. They are especially important in many research areas in computer science like image segmentation, data mining, clustering, network routing, and image capturing. If the role of vertices and edges are uncertain, having two opposite effects, positive and negative, then bipolar fuzzy graphs always play an important factor. In this paper, some important results on different types of operations of bipolar fuzzy graphs are improved. First, we explain some important theorems about the degree of composition, tensor product, and normal product of two bipolar fuzzy graphs using examples.

1. Introduction

Graph theory is connected with the various objects under consideration in this paper. In 1965, Zadeh [18] first replaced the classical set by a fuzzy set, which gives more exactness in both theory and in many real life applications including computer science, network routing, operation research, wireless sensor network, medical science, water and electricity connection in a town, engineering, etc. In 1975, Rosenfeld [15] established the notion of a fuzzy graph, which has many applications in several fields. Mordeson [9] explained many operations on fuzzy graphs. Different types of end nodes in fuzzy graphs and their properties are discussed in [1]. Many indices with their properties in fuzzy graphs and their real life applications in human trafficking and illegal immigration in network are described in [2] and [3], respectively. Domination integrity and its application to fuzzy graphs are explained in [6]. Mathew and Sunitha [8] introduced different types of arcs in fuzzy graphs.

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Due to the existence of bipolarity of a given set, Zhang [19] initiated bipolar fuzzy sets. Yang et al. [17] established generalized bipolar fuzzy graphs and Ghorai and Pal [4] introduced generalized regular bipolar fuzzy graphs. Karunambigai et al. [5] introduced domination in bipolar fuzzy graph. Different types of connectivity in bipolar fuzzy graph are discussed in [7]. Singh et al. [16] defined many lattices in bipolar fuzzy graphs.

Poulik and Ghorai [11] initiated different types of nodes in bipolar fuzzy graphs and showed that they could be applied to research in wireless networks. Several types of connectivity indices and degrees of vertices in bipolar fuzzy graphs are explained in [10, 12]. Poulik et al. [13] initiated many operations on interval-valued fuzzy graphs and applications. Rashmanlou et al. [14] introduced the notion of the product of bipolar fuzzy graphs and their degrees. Theorems 4.1, 5.1, and 6.1 in [14] are not true in general. The main purpose of this paper is to establish the generalized version of all the results and explain them using examples.

2. Preliminaries

Here, some basic definitions of graphs are recalled. These are used later in the paper.

Definition 2.1. [19] Let X be a non-void set. Then we call a set $A = \{(x, \mu_A^P(x), \mu_A^N(x)) : x \in X\}$ a bipolar fuzzy set on X, where $\mu_A^P : X \to [0,1]$ and $\mu_A^N : X \to [-1,0]$ are mappings.

If V is a given set, then an equivalence relation $\sim$ on a set $V \times V - \{(x,x) | x \in V\}$ is defined as $(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow$ either $(x_1, y_1) = (x_2, y_2)$ or $x_1 = y_2$, $x_2 = y_1$. The quotient set produced in this way is denoted by $\widetilde{V^2}$ and the equivalent class which contains the element (x, y) is denoted by xy or yx.

The definition of generalized bipolar fuzzy graph is given below.

Definition 2.2. [17] If $G^* = (V, E)$ is a graph, then a bipolar fuzzy graph is a pair $G = (V, A, B)$, where $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy set in $\widetilde{V^2}$ and $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set in V such that $\mu_B^P(xy) \leqslant \min\{\mu_A^P(x), \mu_A^P(y)\}$ for all $xy \in \widetilde{V^2}$, $\mu_B^N(xy) \geqslant \max\{\mu_A^N(x), \mu_A^N(y)\}$ for all $xy \in \widetilde{V^2}$ and $\mu_B^P(xy) = \mu_B^N(xy) = 0$ for all $xy \in (\widetilde{V^2} - E)$.

Definition 2.3. [4] Let v be a vertex in a bipolar fuzzy graph G. Then the degree of v in G is denoted by $deg(v) = (deg^P(v), deg^N(v))$, where $deg^P(v) = \sum_{\substack{v \neq u \\ uv \in E}} \mu_B^P(uv)$ and $deg^N(v) = \sum_{\substack{v \neq u \\ uv \in E}} \mu_B^N(uv)$.

The definition of composition of two bipolar fuzzy graphs is given below.

Definition 2.4. The composition of two bipolar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ of $G_1^* = (V_1, E_1)$ and $G_2 = (V_2, A_2, B_2)$ of $G_2^* = (V_2, E_2)$ is denoted by $G_1[G_2] = (V, A_1 \circ A_2, B_1 \circ B_2)$ of $G^* = (V_1 \times V_2, E)$ and is defined as

(i) $(\mu_{A_1}^P \circ \mu_{A_2}^P)(a_1, b_1) = \min\{\mu_{A_1}^P(a_1), \mu_{A_2}^P(b_1)\}$
$(\mu_{A_1}^N \circ \mu_{A_2}^N)(a_1, b_1) = \max\{\mu_{A_1}^N(a_1), \mu_{A_2}^N(b_1)\}$ for all $(a_1, b_1) \in V$,

(ii) $(\mu_{B_1}^P \circ \mu_{B_2}^P)((a_1, b_1)(a_1, b_2)) = \min\{\mu_{A_1}^P(a_1), \mu_{B_2}^P(b_1b_2)\}$
$(\mu_{B_1}^N \circ \mu_{B_2}^N)((a_1, b_1)(a_1, b_2)) = \max\{\mu_{A_1}^N(a_1), \mu_{B_2}^N(b_1b_2)\}$ for all $a_1 \in V_1$ and $b_1b_2 \in E_2$,

(iii) $(\mu_{B_1}^P \circ \mu_{B_2}^P)((a_1, b_1)(a_2, b_1)) = \min\{\mu_{B_1}^P(a_1a_2), \mu_{A_2}^P(b_1)\}$
$(\mu_{B_1}^N \circ \mu_{B_2}^N)((a_1, b_1)(a_2, b_1)) = \max\{\mu_{B_1}^N(a_1a_2), \mu_{A_2}^N(b_1)\}$ for all $a_1a_2 \in E_1$ and $b_1 \in V_2$,

(iv) $(\mu_{B_1}^P \circ \mu_{B_2}^P)((a_1, b_1)(a_2, b_2)) = \min\{\mu_{A_2}^P(b_1), \mu_{A_2}^P(b_2), \mu_{B_1}^P(a_1a_2)\}$
$(\mu_{B_1}^N \circ \mu_{B_2}^N)((a_1, b_1)(a_2, b_2)) = \max\{\mu_{A_2}^N(b_1), \mu_{A_2}^N(b_2), \mu_{B_1}^N(a_1a_2)\}$
for all $(a_1, b_1)(a_2, b_2) \in E^0 - E$, where $V = V_1 \times V_2$ and
$E^0 = E \cup \{(a_1, b_1)(a_2, b_2) | a_1a_2 \in E_1, b_1 \neq b_2\}$,
$E = \{(a_1, b_1)(a_1, b_2) | a_1 \in V_1, b_1b_2 \in E_2\} \cup \{(a_1, b_1)(a_2, b_1) | a_2 \in V_2, a_1a_2 \in E_1\}$.

The definition of normal product of two bipolar fuzzy graphs is given below.

Definition 2.5. [14] The normal product of two bipolar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ of $G_1^* = (V_1, E_1)$ and $G_2 = (V_2, A_2, B_2)$ of $G_2^* = (V_2, E_2)$ is denoted by $G_1 \bullet G_2 = (V, A_1 \bullet A_2, B_1 \bullet B_2)$ of $G^* = (V_1 \times V_2, E)$ and is defined as

(i) $(\mu_{A_1}^P \bullet \mu_{A_2}^P)(a_1, b_1) = \min\{\mu_{A_1}^P(a_1), \mu_{A_2}^P(b_1)\}$
$(\mu_{A_1}^N \bullet \mu_{A_2}^N)(a_1, b_1) = \max\{\mu_{A_1}^N(a_1), \mu_{A_2}^N(b_1)\}$, for all $(a_1, b_1) \in V_1 \times V_2$,

(ii) $(\mu_{B_1}^P \bullet \mu_{B_2}^P)((a_1, b_1)(a_1, b_2)) = \min\{\mu_{A_1}^P(a_1), \mu_{B_2}^P(b_1b_2)\}$
$(\mu_{B_1}^N \bullet \mu_{B_2}^N)((a_1, b_1)(a_1, b_2)) = \max\{\mu_{A_1}^N(a_1), \mu_{B_2}^N(b_1b_2)\}$ for all $a_1 \in V_1$ and $b_1b_2 \in E_2$,

(iii) $(\mu_{B_1}^P \bullet \mu_{B_2}^P)((a_1, b_1)(a_2, b_1)) = \min\{\mu_{B_1}^P(a_1a_2), \mu_{A_2}^P(b_1)\}$
$(\mu_{B_1}^N \bullet \mu_{B_2}^N)((a_1, b_1)(a_2, b_1)) = \max\{\mu_{B_1}^N(a_1a_2), \mu_{A_2}^N(b_1)\}$ for all $a_1a_2 \in E_1$ and $b_1 \in V_2$,

(iv) $(\mu_{B_1}^P \bullet \mu_{B_2}^P)((a_1, b_1)(a_2, b_2)) = \min\{\mu_{B_1}^P(a_1a_2), \mu_{B_2}^P(b_1b_2)\}$,
$(\mu_{B_1}^N \bullet \mu_{B_2}^N)((a_1, b_1)(a_2, b_2)) = \max\{\mu_{B_1}^N(a_1a_2), \mu_{B_2}^N(b_1b_2)\}$
for all $a_1b_1 \in E_1$ and $a_2b_2 \in E_2$, where $V = V_1 \times V_2$ and
$E = \{(ab_1)(ab_2) | a \in V_1, b_1b_2 \in E_2\} \cup \{(a_1b)(a_2b) | a_1a_2 \in E_1, b \in V_2\} \cup \{(a_1, b_1)(a_2, b_2) |$
$a_1a_2 \in E_1, b_1b_2 \in E_2\}$.

The definition of tensor product of two bipolar fuzzy graphs is given below.

Definition 2.6. [14] The tensor product of two bipolar fuzzy graphs $G_1 = (V_1, A_1, B_1)$ of $G_1^* = (V_1, E_1)$ and $G_2 = (V_2, A_2, B_2)$ of $G_2^* = (V_2, E_2)$ is denoted by $G_1 \otimes G_2 = (V, A_1 \otimes A_2, B_1 \otimes B_2)$ of $G^* = (V_1 \times V_2, E)$ and is defined as

(i) $(\mu_{A_1}^P \otimes \mu_{A_2}^P)(a_1, b_1) = \min\{\mu_{A_1}^P(a_1), \mu_{A_2}^P(b_1)\}$
$(\mu_{A_1}^N \otimes \mu_{A_2}^N)(a_1, b_1) = \max\{\mu_{A_1}^N(a_1), \mu_{A_2}^N(b_1)\}$ for all $(a_1, b_1) \in V_1 \times V_2$,

(ii) $(\mu_{B_1}^P \otimes \mu_{B_2}^P)((a_1, b_1)(a_2, b_2)) = \min\{\mu_{B_1}^P(a_1a_2), \mu_{B_2}^P(b_1b_2\}$,
$(\mu_{B_1}^N \otimes \mu_{B_2}^N)((a_1, b_1)(a_2, b_2)) = \max\{\mu_{B_1}^N(a_1a_2), \mu_{B_2}^N(b_1b_2)\}$ for all $a_1a_2 \in E_1$ and
$b_1b_2 \in E_2$, where $V = V_1 \times V_2$ and $E = \{(a_1, b_1)(a_2, b_2) | a_1a_2 \in E_1, b_1b_2 \in E_2\}$.

The definition of degree of tensor product of two bipolar fuzzy graphs is given below.

Definition 2.7. [14] The degree of a vertex (a_1, b_1) in the tensor product of two bipolar fuzzy graphs G_1 and G_2 is denoted by $d_{G_1 \otimes G_2}(a_1, b_1) = (d_{G_1 \otimes G_2}^P(a_1, b_1), d_{G_1 \otimes G_2}^N(a_1, b_1))$ and is defined as

$$\begin{aligned} d_{G_1 \otimes G_2}^P(a_1, b_1) &= \sum (\mu_{B_1}^P \otimes \mu_{B_2}^P)((a_1, b_2)(a_2, b_1)) \\ &= \sum_{a_1a_2 \in E_1 b_1b_2 \in E_2} \mu_{B_1}^P(a_1a_2) \wedge \mu_{B_2}^P(b_1b_2), \end{aligned}$$

and

$$\begin{aligned} d_{G_1 \otimes G_2}^N(a_1, b_1) &= \sum (\mu_{B_1}^N \otimes \mu_{B_2}^N)((a_1, b_2)(a_2, b_1)) \\ &= \sum_{\substack{a_1a_2 \in E_1 \\ b_1b_2 \in E_2}} \mu_{B_1}^N(a_1a_2) \vee \mu_{B_2}^N(b_1b_2). \end{aligned}$$

The definition of degree of normal product of two bipolar fuzzy graphs is given below.

Definition 2.8. [14] The degree of a vertex (a_1, b_1) in a normal product of two bipolar fuzzy graphs G_1 and G_2 is denoted by $d_{G_1 \bullet G_2}(a_1, b_1) =$

$(d^P_{G_1 \bullet G_2}(a_1, b_1), d^N_{G_1 \bullet G_2}(a_1, b_1))$ and is defined as

$$\begin{aligned}
d^P_{G_1 \bullet G_2}(a_1, b_1) &= \sum_{((a_1,b_2)(a_2,b_1)) \in E} (\mu^P_{B_1} \bullet \mu^P_{B_2})((a_1, b_2)(a_2, b_1)) \\
&= \sum_{\substack{a_1 = a_2 \\ b_1 b_2 \in E_2}} \mu^P_{A_1}(a_1) \wedge \mu^P_{B_2}(b_1 b_2) + \sum_{\substack{b_1 = b_2 \\ a_1 a_2 \in E_1}} \mu^P_{A_2}(b_1) \\
&\wedge \quad \mu^P_{B_1}(a_1 a_2) + \sum_{\substack{a_1 a_2 \in E_1 \\ b_1 b_2 \in E_2}} \mu^P_{B_1}(a_1 a_2) \wedge \mu^P_{B_2}(b_1 b_2),
\end{aligned}$$

$$\begin{aligned}
d^N_{G_1 \bullet G_2}(a_1, b_1) &= \sum_{((a_1,b_2)(a_2,b_1)) \in E} (\mu^N_{B_1} \bullet \mu^N_{B_2})((a_1, b_2)(a_2, b_1)) \\
&= \sum_{\substack{a_1 = a_2 \\ b_1 b_2 \in E_2}} \mu^N_{A_1}(a_1) \vee \mu^N_{B_2}(b_1 b_2) + \sum_{\substack{b_1 = b_2 \\ a_1 a_2 \in E_1}} \mu^N_{A_2}(b_1) \\
&\vee \quad \mu^N_{B_1}(a_1 a_2) + \sum_{\substack{a_1 a_2 \in E_1 \\ b_1 b_2 \in E_2}} \mu^N_{B_1}(a_1 a_2) \vee \mu^N_{B_2}(b_1 b_2).
\end{aligned}$$

The definition of degree of composition of two bipolar fuzzy graphs is given below.

Definition 2.9. [14] The degree of a vertex (a_1, b_1) in the composition of two bipolar fuzzy graphs G_1 and G_2 is denoted by $d_{G_1[G_2]}(a_1, b_1) = (d^P_{G_1[G_2]}(a_1, b_1), d^N_{G_1[G_2]}(a_1, b_1))$ and is defined as

$$\begin{aligned}
d^P_{G_1[G_2]}(a_1, b_1) &= \sum_{(a_1,b_2)(a_2,b_1) \in E} (\mu^P_{B_1} \circ \mu^P_{B_2})((a_1, b_2)(a_2, b_1)) \\
&= \sum_{\substack{a_1 = a_2 \\ b_1 b_2 \in E_2}} \mu^P_{A_1}(a_1) \wedge \mu^P_{B_2}(b_1 b_2) + \sum_{\substack{b_1 = b_2 \\ a_1 a_2 \in E_1}} \mu^P_{A_2}(b_1) \\
&\wedge \quad \mu^P_{B_1}(a_1 a_2) + \sum_{\substack{b_1 \neq b_2 \\ a_1 a_2 \in E_1}} \mu^P_{A_2}(b_1) \wedge \mu^P_{A_2}(b_2) \wedge \mu^P_{B_1}(a_1 a_2),
\end{aligned}$$

$$\begin{aligned}
d^N_{G_1[G_2]}(a_1, b_1) &= \sum_{(a_1,b_2)(a_2,b_1) \in E} (\mu^N_{B_1} \circ \mu^N_{B_2})((a_1, b_2)(a_2, b_1)) \\
&= \sum_{\substack{a_1 = a_2 \\ b_1 b_2 \in E_2}} \mu^N_{A_1}(a_1) \vee \mu^N_{B_2}(b_1 b_2) + \sum_{\substack{b_1 = b_2 \\ a_1 a_2 \in E_1}} \mu^N_{A_2}(b_1) \\
&\vee \quad \mu^N_{B_1}(a_1 a_2) + \sum_{\substack{b_1 \neq b_2 \\ a_1 a_2 \in E_1}} \mu^N_{A_2}(b_1) \vee \mu^N_{A_2}(b_2) \vee \mu^N_{B_1}(a_1 a_2).
\end{aligned}$$

3. Counterexamples

Here we recall Theorems 4.1, 5.1, and 6.1, which are given in [14] and then show by counterexample that these theorems are not true in general.

Theorem 3.1. *(Theorem 4.1 in* [14]*) Let* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *be two bipolar fuzzy graphs. If* $\mu^P_{A_1} \geq \mu^P_{B_2}$, $\mu^N_{A_1} \leq \mu^N_{B_2}$, $\mu^P_{A_2} \geq \mu^P_{B_1}$, *and* $\mu^N_{A_2} \leq \mu^N_{B_1}$, *then*
$d_{G_1[G_2]}(u_1, u_2) = |V_2| d_{G_1}(u_1) + d_{G_2}(u_2)$ *for all* $(u_1, u_2) \in V_1 \times V_2$.

Theorem 3.1 is not true in general, which is shown in Example 3.2.

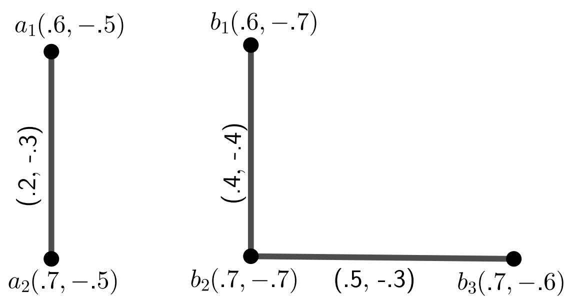

Figure 1. Two bipolar fuzzy graphs G_1 and G_2.

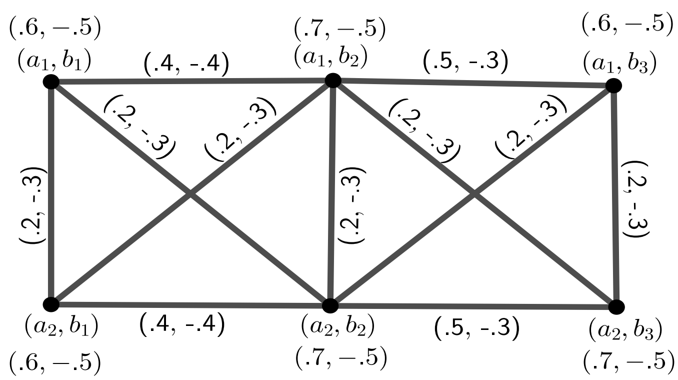

Figure 2. $G_1[G_2]$ (Composition of G_1 and G_2 of Figure 1)).

Example 3.2. *We consider the two bipolar fuzzy graphs* G_1 *and* G_2 *from Figure 1. The composition of* G_1 *and* G_2 *is* $G_1[G_2]$ *(see Figure 2). Also, we assume that the membership value of all vertices of* $G_1[G_2]$ *are the same as in* G_1 *and* G_2*, that is* $(1, -1)$*. Now by Theorem 4.1 in* [14]*, we have*
$d^P_{G_1[G_2]}(a_1, b_1) = d^P_{G_2}(b_1) + |V_2| d^P_{G_1}(a_1) = 0.4 + 3 \times (0.2) = 1.0$ *and*
$d^N_{G_1[G_2]}(a_1, b_1) = d^N_{G_2}(b_1) + |V_2| d^N_{G_1}(a_1) = -0.4 + 3 \times (-0.3) = -1.3$. *So* $d_{G_1[G_2]}(a_1, b_1) = (1.0, -1.3)$. *But actually,* $d_{G_1[G_2]}(a_1, b_1) = (0.8, -1.0)$. *So Theorem 4.1 in* [14] *is not true.*

Theorem 3.3. *(Theorem 5.1 in* [14]*) Let* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *be two bipolar fuzzy graphs. If* $\mu^P_{B_2} \geq \mu^P_{B_1}$ *and* $\mu^N_{B_2} \leq \mu^N_{B_1}$*, then* $d_{G_1 \otimes G_2}(u_1, u_2) = d_{G_1}(u_1)$ *and if* $\mu^P_{B_1} \geq \mu^P_{B_2}$ *and* $\mu^N_{B_1} \leq \mu^N_{B_2}$*, then* $d_{G_1 \otimes G_2}(u_1, u_2) = d_{G_1}(u_2)$.

Theorem 3.3 is not true in general. This is explained in Example 3.4.

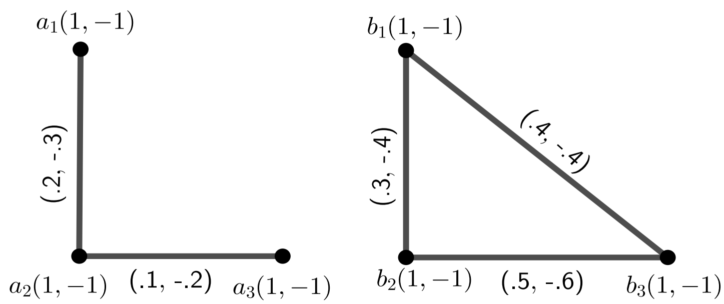

FIGURE 3. Two bipolar fuzzy graphs G_1 and G_2.

Example 3.4. *We consider two bipolar fuzzy graphs* G_1 *and* G_2 *from Figure 3 and their tensor product* $G_1 \otimes G_2$*. This is shown in Figure 4. Also, we assume that the membership value of all vertices of* $G_1 \otimes G_2$ *are the same as in* G_1 *and* G_2*, that is* $(1, -1)$*. Now by Theorem 5.1 in* [14]*, we have*
$d^P_{G_1 \otimes G_2}(a_1, b_1) = d^P_{G_1}(a_1) = 0.2$ *and* $d^N_{G_1 \otimes G_2}(a_1, b_1) = d^N_{G_1}(a_1) = -0.3$.
So $d_{G_1 \otimes G_2}(a, b) = (0.2, -0.3)$.
But actually, $d_{G_1 \otimes G_2}(a, b) = (0.4, -0.6)$. *So Theorem 5.1 in* [14] *is not true.*

Theorem 3.5. *(Theorem 6.1 in* [14]*) Let* $G_1 = (V_1, A_1, B_1)$ *and* $G_2 = (V_2, A_2, B_2)$ *be two bipolar fuzzy graphs. If* $\mu^P_{A_1} \geq \mu^P_{B_2}$, $\mu^N_{A_1} \leq \mu^N_{B_2}$, $\mu^P_{A_2} \geq$

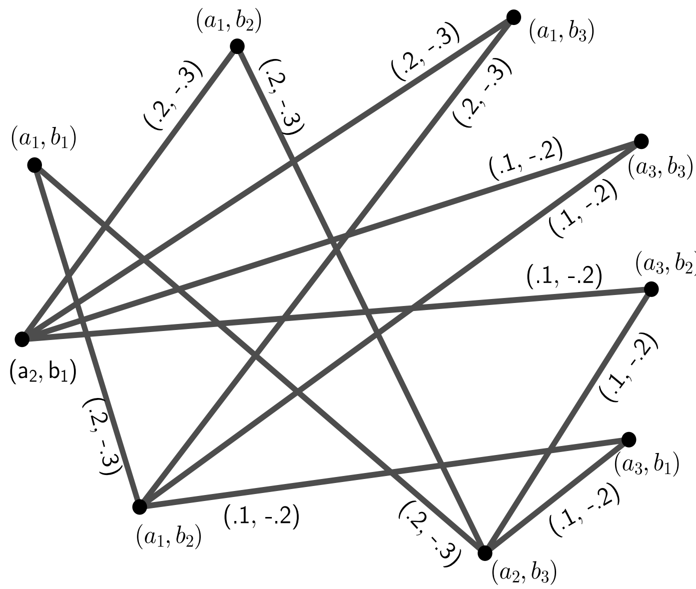

FIGURE 4. $G_1 \otimes G_2$ (Tensor product of G_1 and G_2 of Figure 3).

$\mu^P_{B_1}$, $\mu^N_{A_2} \leq \mu^N_{B_1}$, $\mu^P_{B_1} \leq \mu^P_{B_2}$, *and* $\mu^N_{B_1} \geq \mu^N_{B_2}$, *then* $d_{G_1 \bullet G_2} = |V_2| d_{G_1}(u_1) + d_{G_2}(u_2)$.

Theorem 3.5 is not true in general, which is shown in Example 3.6.

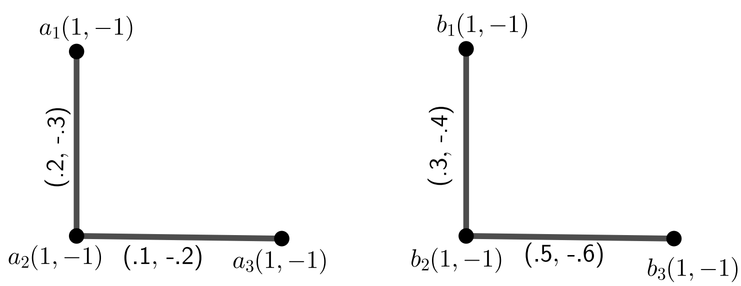

FIGURE 5. Two bipolar fuzzy graphs G_1 and G_2.

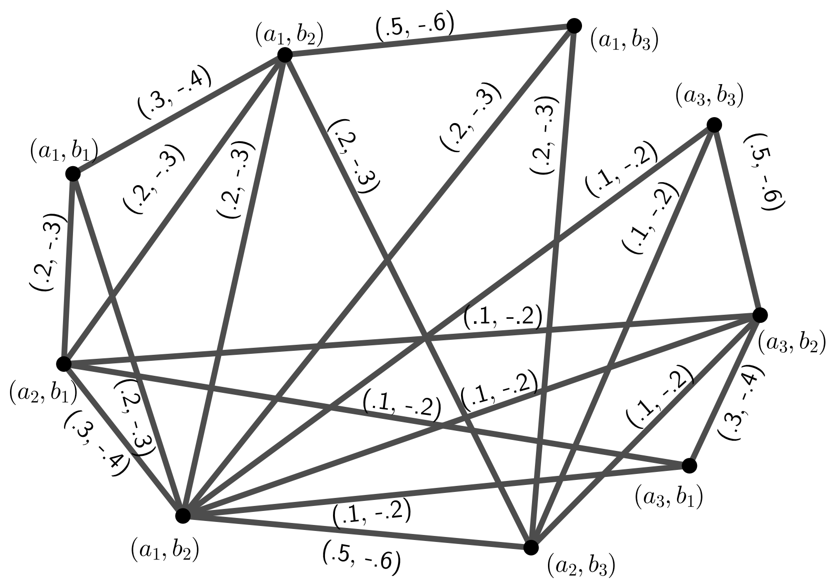

FIGURE 6. $G_1 \bullet G_2$ (Normal product of G_1 and G_2 of Figure 5).

Example 3.6. *We consider two bipolar fuzzy graphs G_1 and G_2 of Figure 5 and the normal product of G_1 and G_2 is $G_1 \bullet G_2$ (see Figure 6). Also, we assume that the membership value of all vertices of $G_1 \bullet G_2$ are the same as in G_1 and G_2, that is $(1, -1)$. Now by Theorem 6.1 in* [14], *we have* $d^P_{G_1 \bullet G_2}(a_2, b_3) = d^P_{G_2}(b_3) + |V_2| d^P_{G_1}(a_2) = 0.5 + 3 \times (0.2 + 0.1) = 1.4$ *and* $d^N_{G_1 \bullet G_2}(a_2, b_3) = d^N_{G_2}(b_3) + |V_2| d^N_{G_1}(a_2) = -0.6 + 3 \times (-0.3 - 0.2) = -2.1$. *So* $d_{G_1 \bullet G_2}(a_2, b_3) = (1.4, -2.1)$. *But actually,* $d_{G_1 \bullet G_2}(a, b) = (1.1, -1.6)$. *So Theorem 6.1 in* [14] *is not true.*

4. Main Results

In this section, we provide more generalized results of Theorems 4.1, 5.1, and 6.1 of [14]. These results are verified with examples also.

Theorem 4.1. *(Correction of Theorem 4.1 of* [14]*.) Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two bipolar fuzzy graphs. Let $a \in V_1$ and assume G_1 has m edges incident with a which are aa_1, aa_2, …, aa_m and $b \in V_2$ and assume G_2 has n edges incident with b which are bb_1, bb_2, …, bb_n. If*

$\mu^P_{B_2} \leq \mu^P_{A_1}$, $\mu^N_{B_2} \geq \mu^N_{A_1}$, $\mu^P_{B_1} \leq \mu^P_{A_2}$, and $\mu^N_{B_1} \geq \mu^N_{A_2}$, then $d_{G_1[G_2]}(a,b) = d_{G_2}(b) + (n+1)d_{G_1}(a)$.

Proof.

$$
\begin{aligned}
& d^P_{G_1[G_2]}(a,b) \\
&= \sum_{((aa_i)(bb_j))\in E} (\mu^P_{B_1} \circ \mu^P_{B_2})((aa_i)(bb_j)) \\
&= \sum_{a=a_i bb_j \in E_2} \mu^P_{A_1}(a) \wedge \mu^P_{B_2}(bb_j) + \sum_{b=b_j aa_i \in E_1} \mu^P_{A_2}(b) \wedge \mu^P_{B_1}(aa_i) \\
&+ \sum_{aa_i \in E_1 bb_j \in E_2} \mu^P_{A_2}(b) \wedge \mu^P_{A_2}(b_j) \wedge \mu^P_{B_1}(aa_i) \\
&= \sum_{j=1}^{n} \mu^P_{B_2}(bb_j) + \sum_{i=1}^{m} \mu^P_{B_1}(aa_i) + \big[\{\mu^P_{A_2}(b) \wedge \mu^P_{A_2}(b_1) \wedge \mu^P_{B_1}(aa_1) \\
&+ \mu^P_{A_2}(b) \wedge \mu^P_{A_2}(b_2) \wedge \mu^P_{B_1}(aa_1) + \cdots + \mu^P_{A_2}(b) \wedge \mu^P_{A_2}(b_n) \wedge \mu^P_{B_1}(aa_1)\} \\
&+ \{\mu^P_{A_2}(b) \wedge \mu^P_{A_2}(b_1) \wedge \mu^P_{B_1}(aa_2) + \mu^P_{A_2}(b) \wedge \mu^P_{A_2}(b_2) \wedge \mu^P_{B_1}(aa_2) \\
&+ \cdots + \mu^P_{A_2}(b) \wedge \mu^P_{A_2}(b_n) \wedge \mu^P_{B_1}(aa_2)\} + \cdots \\
&+ \{\mu^P_{A_2}(b) \wedge \mu^P_{A_2}(b_1) \wedge \mu^P_{B_1}(aa_m) + \mu^P_{A_2}(b) \wedge \mu^P_{A_2}(b_2) \wedge \mu^P_{B_1}(aa_m) \\
&+ \cdots + \mu^P_{A_2}(b) \wedge \mu^P_{A_2}(b_n) \wedge \mu^P_{B_1}(aa_m)\}\big]
\end{aligned}
$$

(since $\mu^P_{B_2} \leq \mu^P_{A_1}$ and $\mu^P_{B_1} \leq \mu^P_{A_2}$)

$$= d^P_{G_2}(b) + d^P_{G_1}(a) + n\big[\mu^P_{B_1}(aa_1 + \mu^P_{B_1}(aa_2) + \cdots + \mu^P_{B_1}(aa_m)\big]$$

(since $\mu^P_{B_2} \leq \mu^P_{A_1}$ and $\mu^P_{B_1} \leq \mu^P_{A_2}$)

$$= d^P_{G_2}(b) + d^P_{G_1}(a) + nd^P_{G_1}(a) = d^P_{G_2}(b) + (n+1)d^P_{G_1}(a),$$

and

$$
\begin{aligned}
& d^N_{G_1[G_2]}(a,b) \\
&= \sum_{(aa_i)(bb_j)\in E} (\mu^N_{B_1} \circ \mu^N_{B_2})((aa_i)(bb_j)) \\
&= \sum_{a=a_i bb_j \in E_2} \mu^N_{A_1}(a) \vee \mu^N_{B_2}(bb_j) + \sum_{b=b_j aa_i \in E_1} \mu^N_{A_2}(b) \vee \mu^N_{B_1}(aa_i) \\
&+ \sum_{aa_i \in E_1 bb_j \in E_2} \mu^N_{A_2}(b) \vee \mu^N_{A_2}(b_j) \vee \mu^N_{B_1}(aa_i)
\end{aligned}
$$

$$= \sum_{j=1}^{n} \mu^N_{B_2}(bb_j) + \sum_{i=1}^{m} \mu^N_{B_1}(aa_i) + \big[\{\mu^N_{A_2}(b) \vee \mu^N_{A_2}(b_1) \vee \mu^N_{B_1}(aa_1)$$
$$+ \mu^N_{A_2}(b) \vee \mu^N_{A_2}(b_2) \vee \mu^N_{B_1}(aa_1) + \cdots$$
$$+ \mu^N_{A_2}(b) \vee \mu^N_{A_2}(b_n) \vee \mu^N_{B_1}(aa_1)\} + \{\mu^N_{A_2}(b) \vee \mu^N_{A_2}(b_1) \vee \mu^N_{B_1}(aa_2)$$
$$+ \mu^N_{A_2}(b) \vee \mu^N_{A_2}(b_2) \vee \mu^N_{B_1}(aa_2) + \cdots$$
$$+ \mu^N_{A_2}(b) \vee \mu^N_{A_2}(b_n) \vee \mu^N_{B_1}(aa_2)\} + \cdots + \{\mu^N_{A_2}(b) \vee \mu^N_{A_2}(b_1) \vee \mu^N_{B_1}(aa_m)$$
$$+ \mu^N_{A_2}(b) \vee \mu^N_{A_2}(b_2) \vee \mu^N_{B_1}(aa_m) + \cdots + \mu^N_{A_2}(b) \vee \mu^N_{A_2}(b_n) \vee \mu^N_{B_1}(aa_m)\}\big]$$

(since $\mu^N_{B_2} \geq \mu^N_{A_1}$ and $\mu^N_{B_1} \geq \mu^N_{A_2}$)

$$= d^N_{G_2}(b) + d^N_{G_1}(a) + n\big[\mu^N_{G_1}(aa_1 + \mu^N_{B_1}(aa_2) + \cdots + \mu^N_{B_1}(aa_m)\big]$$

(since $\mu^N_{B_2} \geq \mu^N_{A_1}$ and $\mu^N_{B_1} \geq \mu^N_{A_2}$)

$$= d^N_{G_2}(b) + d^N_{G_1}(a) + nd^N_{G_1}(a) = d^N_{G_2}(b) + (n+1)d^N_{G_1}(a).$$

So

$$\begin{aligned} d_{G_1[G_2]}(a,b) &= (d^P_{G_1[G_2]}(a,b), d^N_{G_1[G_2]}(a,b)) \\ &= (d^P_{G_2}(b) + (n+1)d^P_{G_1}(a), d^N_{G_2}(b) + (n+1)d^N_{G_1}(a)) \\ &= (d^P_{G_2}(b), (d^N_{G_2}(b)) + (n+1)(d^P_{G_1}(a), d^N_{G_1}(a)) \\ &= d_{G_2}(b) + (n+1)d_{G_1}(a). \end{aligned}$$

□

Example 4.2. *Now using Theorem 4.1 from Figure 2, we have*

$$d^P_{G_1[G_2]}(a_1, b_1) = d^P_{G_2}(b_1) + (1+1)d^P_{G_1}(a_1) = 0.4 + 2 \times (0.2) = 0.8,$$

because the number of edges incident in b_1 in G_2 is 1.

$$d^N_{G_1[G_2]}(a_1, b_1) = d^N_{G_2}(b_1) + (1+1)d^N_{G_1}(a_1) = -0.4 + 2 \times (-0.3) = -1.0,$$

because the number of edges incident in b_1 in G_2 is 1. So $d_{G_1[G_2]}(a_1, b_1) = (0.8, -1.0)$ satisfies Theorem 4.1. Similarly the degree of other vertices of $G_1[G_2]$ satisfy Theorem 4.1.

Theorem 4.3. *(Correction of Theorem 5.1 of* [14]*.) Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two bipolar fuzzy graphs. Let $a \in V_1$ and assume G_1 has m edges aa_1, aa_2, …, aa_m incident with a. Let $b \in V_2$ and assume G_2 has n edges bb_1, bb_2, …, bb_n incident with b.*

(i) *If $\mu^P_{B_2} \geq \mu^P_{B_1}$ and $\mu^N_{B_2} \leq \mu^N_{B_1}$ for all edges in G_1 and G_2, then $d_{G_1 \otimes G_2}(a,b) = n\ d_{G_1}(a)$.*

(ii) *If* $\mu^P_{B_2} \le \mu^P_{B_1}$ *and* $\mu^N_{B_2} \ge \mu^N_{B_1}$ *for all edges in* G_1 *and* G_2*, then* $d_{G_1 \otimes G_2}(a,b) = m\ d_{G_2}(b)$.

Proof. (i) Let $\mu^P_{B_2} \ge \mu^P_{B_1}$ and $\mu^N_{B_2} \le \mu^N_{B_1}$. So $\mu^P_{B_2}(bb_j) \ge \mu^P_{B_1}(aa_i)$ and $\mu^N_{B_2}(bb_j) \le \mu^N_{B_1}(aa_i)$ for $1 \le i \le m$ and $1 \le j \le n$.

Now

$$\begin{aligned}
&d^P_{G_1 \otimes G_2}(a,b)\\
&= \{\mu^P_{B_1}(aa_1) \wedge \mu^P_{B_2}(bb_1) + \mu^P_{B_1}(aa_2) \wedge \mu^P_{B_2}(bb_1)\\
&+ \cdots + \mu^P_{B_1}(aa_m) \wedge \mu^P_{B_2}(bb_1)\} + \{\mu^P_{B_1}(aa_1) \wedge \mu^P_{B_2}(bb_2)\\
&+ \mu^P_{B_1}(aa_2) \wedge \mu^P_{B_2}(bb_2) + \cdots + \mu^P_{B_1}(aa_m) \wedge \mu^P_{B_2}(bb_2)\}\\
&+ \cdots + \{\mu^P_{B_1}(aa_1) \wedge \mu^P_{B_2}(bb_n) + \mu^P_{B_1}(aa_2) \wedge \mu^P_{B_2}(bb_n)\\
&+ \cdots + \mu^P_{B_1}(aa_m) \wedge \mu^P_{B_2}(bb_n)\}\\
&= n\{\mu^P_{B_1}(aa_1) + \mu^P_{B_1}(aa_2) + \cdots + \mu^P_{B_1}(aa_m)\} = nd^P_{G_1}(a)
\end{aligned}$$

and

$$\begin{aligned}
&d^N_{G_1 \otimes G_2}(a,b)\\
&= \{\mu^N_{B_1}(aa_1) \vee \mu^N_{B_2}(bb_1) + \mu^N_{B_1}(aa_2) \vee \mu^N_{B_2}(bb_1) + \cdots\\
&+ \mu^N_{B_1}(aa_m) \vee \mu^N_{B_2}(bb_1)\} + \{\mu^N_{B_1}(aa_1) \vee \mu^N_{B_2}(bb_2) + \mu^N_{B_1}(aa_2) \vee \mu^N_{B_2}(bb_2)\\
&+ \cdots + \mu^N_{B_1}(aa_m) \vee \mu^N_{B_2}(bb_2)\} + \cdots\\
&+ \{\mu^N_{B_1}(aa_1) \vee \mu^N_{B_2}(bb_n) + \mu^N_{B_1}(aa_2) \vee \mu^N_{B_2}(bb_n)\\
&+ \cdots + \mu^N_{B_1}(aa_m) \vee \mu^N_{B_2}(bb_n)\}\\
&= n\{\mu^N_{B_1}(aa_1) + \mu^N_{B_1}(aa_2) + \cdots + \mu^N_{B_1}(aa_m)\} = nd^N_{G_1}(a).
\end{aligned}$$

So

$$\begin{aligned}
d_{G_1 \otimes G_2}(a,b) &= (d^P_{G_1 \otimes G_2}(a,b), d^N_{G_1 \otimes G_2}(a,b))\\
&= (nd^P_{G_1}(a), nd^N_{G_1}(a))\\
&= n(d^P_{B_1}(a), d^N_{B_1}(a))\\
&= nd_{G_1}(a).
\end{aligned}$$

Similarly (ii) can be proved. □

Example 4.4. *Now using Theorem 4.3 from Figure 4, we have*

$$d^P_{G_1 \otimes G_2}(a_1,b_1) = 2 \times d^P_{G_1}(a_1) = 2 \times (0.2) = 0.4,$$

because the number of edges incident in b_1 *in* G_2 *is 2.*

$$d^N_{G_1 \otimes G_2}(a_1,b_1) = 2 \times d^P_{G_1}(a_1) = 2 \times (-0.3) = -0.6,$$

because the number of edges incident in b_1 in G_2 is 2. So $d_{G_1 \otimes G_2}(a_1, b_1) = (0.4, -0.6)$ satisfies Theorem 4.3. Similarly the degree of other vertices of $G_1 \otimes G_2$ satisfy Theorem 4.3.

Theorem 4.5. *(Correction of Theorem 6.1 of* [14]*.) Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two bipolar fuzzy graphs. Let $a \in V_1$ and assume G_1 has m edges aa_1, aa_2, $\ldots$, aa_m incident with a. Let $b \in V_2$ and assume G_2 has n edges bb_1, bb_2, $\ldots$, bb_n incident with b. If $\mu^P_{A_1} \geq \mu^P_{B_2}$, $\mu^N_{A_1} \leq \mu^B_{B_2}$, $\mu^P_{A_2} \geq \mu^P_{B_1}$, $\mu^N_{A_2} \leq \mu^N_{B_1}$, $\mu^P_{B_1} \leq \mu^P_{B_2}$, and $\mu^N_{B_1} \geq \mu^N_{B_2}$, then $d_{G_1 \bullet G_2}(a, b) = d_{G_2}(b) + (n+1)d_{G_1}(a)$.*

Proof.

$$
\begin{aligned}
&d^P_{G_1 \bullet G_2}(a, b) \\
&= \sum_{((aa_i)(bb_j)) \in E} (\mu^P_{B_1} \bullet \mu^P_{B_2})((aa_i)(bb_j)) \\
&= \sum_{a = a_i bb_j \in E_2} \mu^P_{A_1}(a) \wedge \mu^P_{B_2}(bb_j) + \sum_{b = b_j aa_i \in E_1} \mu^P_{A_2}(b) \wedge \mu^P_{B_1}(aa_i) \\
&+ \sum_{aa_i \in E_1 bb_j \in E_2} \mu^P_{B_1}(aa_i) \wedge \mu^P_{B_2}(bb_j) \\
&= \sum_{j=1}^{n} \mu^P_{B_2}(bb_j) \\
&+ \sum_{i=1}^{m} \mu^P_{B_1}(aa_i) + \big[\{\mu^P_{B_1}(aa_1) \wedge \mu^P_{B_2}(bb_1) + \mu^P_{B_1}(aa_2) \wedge \mu^P_{B_2}(bb_1) \\
&+ \cdots + \mu^P_{B_1}(aa_m) \wedge \mu^P_{B_2}(bb_1)\} + \{\mu^P_{B_1}(aa_1) \wedge \mu^P_{B_2}(bb_2) \\
&+ \mu^P_{B_1}(aa_2) \wedge \mu^P_{B_2}(bb_2) + \cdots + \mu^P_{B_1}(aa_m) \wedge \mu^P_{B_2}(bb_2)\} \\
&+ \cdots + \{\mu^P_{B_1}(aa_1) \wedge \mu^P_{B_2}(bb_n) + \mu^P_{B_1}(aa_2) \wedge \mu^P_{B_2}(bb_n) \\
&+ \cdots + \mu^P_{B_1}(aa_m) \wedge \mu^P_{B_2}(bb_n)\}\big]
\end{aligned}
$$

(since $\mu^P_{A_1} \geq \mu^P_{B_2}$ and $\mu^P_{A_2} \geq \mu^P_{B_1}$)

$$= d^P_{G_2}(b) + d^P_{G_1}(a) + n\{\mu^P_{G_1}(aa_1) + \mu^P_{B_1}(aa_2) + \cdots + \mu^P_{B_1}(aa_m)\}$$

(since $\mu^P_{B_1} \leq \mu^P_{B_2}$)

$$= d^P_{G_2}(b) + d^P_{G_1}(a) + nd^P_{G_1}(a) = d^P_{G_2}(b) + (n+1)d^P_{G_1}(a),$$

and

$$\begin{aligned}
&d^N_{G_1 \bullet G_2}(a,b)\\
&= \sum_{((aa_i)(bb_j))\in E} (\mu^N_{B_1} \bullet \mu^N_{B_2})((aa_i)(bb_j))\\
&= \sum_{a=a_i bb_j \in E_2} \mu^N_{A_1}(a) \vee \mu^N_{B_2}(bb_j) + \sum_{b=b_j aa_i \in E_1} \mu^N_{A_2}(b) \vee \mu^N_{B_1}(aa_i)\\
&+ \sum_{aa_i \in E_1 bb_j \in E_2} \mu^N_{B_1}(aa_i) \vee \mu^N_{B_2}(bb_j) = \sum_{j=1}^{n} \mu^N_{B_2}(bb_j)\\
&+ \sum_{i=1}^{m} \mu^N_{B_1}(aa_i) + \big[\{\mu^N_{B_1}(aa_1) \vee \mu^N_{B_2}(bb_1) + \mu^N_{B_1}(aa_2) \vee \mu^N_{B_2}(bb_1)\\
&+ \cdots + \mu^N_{B_1}(aa_m) \vee \mu^N_{B_2}(bb_1)\} + \{\mu^N_{B_1}(aa_1) \vee \mu^P_{B_2}(bb_2)\\
&+ \mu^N_{B_1}(aa_2) \vee \mu^N_{B_2}(bb_2) + \cdots + \mu^N_{B_1}(aa_m) \vee \mu^N_{B_2}(bb_2)\}\\
&+ \cdots + \{\mu^N_{B_1}(aa_1) \vee \mu^N_{B_2}(bb_n) + \mu^N_{B_1}(aa_2) \vee \mu^N_{B_2}(bb_n)\\
&+ \cdots + \mu^N_{B_1}(aa_m) \vee \mu^N_{B_2}(bb_n)\}\big]
\end{aligned}$$

(since $\mu^N_{A_1} \leq \mu^B_{B_2}$ and $\mu^N_{A_2} \leq \mu^N_{B_1}$)

$$= d^N_{G_2}(b) + d^N_{G_1}(a) + n\{\mu^N_{B_1}(aa_1) + \mu^N_{B_1}(aa_2) + \cdots + \mu^N_{B_1}(aa_m)\}$$

(since $\mu^N_{B_1} \geq \mu^N_{B_2}$)

$$= d^N_{G_2}(b) + d^N_{G_1}(a) + nd^N_{G_1}(a) = d^N_{G_2}(b) + (n+1)d^N_{G_1}(a).$$

□

Example 4.6. *Now using Theorem 4.5 from Figure 6, we have*

$$d^P_{G_1 \bullet G_2}(a_2, b_3) = d^P_{G_2}(b_3) + (1+1)d^P_{G_1}(a_2) = 0.5 + 2 \times (0.2 + 0.1) = 1.1$$

because the number of edges incident in b_1 in G_2 is 1.

$$d^N_{G_1 \bullet G_2}(a_2, b_3) = d^N_{G_2}(b_3) + (1+1)d^N_{G_1}(a_2) = -0.6 + 2 \times (-0.2 - 0.3) = -1.6$$

because the number of edges incident in b_1 in G_2 is 1. So $d_{G_1 \bullet G_2}(a_2, b_3) = (1.1, -1.6)$ satisfies Theorem 4.5. Similarly the degree of other vertices of $G_1 \bullet G_2$ satisfy Theorem 4.5.

5. Conclusion

Graph theory gives many important results for solving different types of combinatorial problems including algebra, geometry, number theory, optimization, computer science, and operational research. But for the uncertainty of vertices and edges, both in the sense of positive and negative in graph, bipolar fuzzy graph gives more useful results for solving many real life problems. First it has been shown, by examples, that the formulas for calculating degree of vertices in some operations like composition, normal product, and tensor product of two bipolar fuzzy graphs are not true in general. Then we established the updated version of these theorems on degree of a vertex in different types of product of bipolar fuzzy graphs.

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